INCOMPATIBILITY-GOVERNED SINGULARITIES IN LINEAR ELASTICITY WITH DISLOCATIONS

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ABSTRACT. The purpose of this paper is to prove the relation $\text{inc}\varepsilon = \text{Curl}\,\kappa$ relating the elastic strain ε and the contortion tensor κ, related to the density tensor of mesoscopic dislocations. Here, the dislocations are given by a finite family of closed Lipschitz curves in $\Omega \subset \mathbb{R}^3$. Moreover the fields are singular at the dislocations, and in particular the strain is non square integrable. Moreover, the displacement fields shows a constant jump around each isolated dislocation loop. This relation is called after E. Kröner who first derived the same formula for smooth fields at the macroscale.

1. Introduction

Let Ω be a simply-connected smooth and bounded subset of $\mathbb{R}^3$. Let $\mathcal{L}$ be a set of dislocation lines in Ω, and the dislocation density $\Lambda_{\mathcal{L}} \in \mathcal{M}(\Omega, \mathbb{M}^3)$ be given as a Radon measure concentrated in $\mathcal{L}$, defined as

$$\Lambda_{\mathcal{L}} := \tau \otimes B\mathcal{H}^1_{\lfloor\mathcal{L}}, \tag{1}$$

with τ, the tangent vector to $\mathcal{L}$ and $\mathcal{H}^1_{\lfloor\mathcal{L}}$ the one-dimensional Hausdorff measure concentrated in $\mathcal{L}$, and where B stand for the Burgers vector of the line, constant for a given line. Note that by definition, $\text{Div}\,\Lambda^T_{\mathcal{L}} = 0$. Another dislocation density tensor, the contortion, is introduced as follows:

$$\kappa_{\mathcal{L}} := \Lambda_{\mathcal{L}} - \frac{\mathbb{I}_2}{2}\,\text{tr}\Lambda_{\mathcal{L}}.$$

By (1), it is seen that $\text{tr}\Lambda_{\mathcal{L}} = 0$ for edge dislocations, for which the Burgers vector is orthogonal to the line. Therefore, $\kappa_{\mathcal{L}} = \Lambda_{\mathcal{L}}$ for pure edge dislocation loops, that is, planar loops with out-of-plane Burgers vector.

It is well known that as soon as dislocations are present, i.e. as soon as their density is nonvanishing, the strain can not be a symmetric gradient of a vector field. At the macroscopic scale, that is, at a scale where the fields are assumed smooth, it is indeed known that the incompatibility of the elastic strain ε is related to the curl of the contortion tensor κ. Here the contortion is a symmetric tensor related to the macroscopic dislocation density Λ by the relation $\kappa = \Lambda - \frac{\mathbb{I}_2}{2}\,\text{tr}\Lambda$, with Λ the macroscopic dislocation density. The so-called Kröner's identity reads

$$\text{inc}\varepsilon = \text{Curl}\,\kappa. \tag{2}$$

This relation was to the knowledge of the author first introduced by Ekkehart Kröner in [6] (see also [7]) it strictly spoken appeared first in [13] in a simple geometrical setting. The geometrical meaning of the contortion tensor in a differential geometry approach to dislocations has to be emphasized, as discussed in e.g., [5, 7, 11, 17]. In differential geometry as well as in generalized non-Riemannian gravity, such a relation is well-known (see [8, 9]) and is nothing but the condition of teleparallelism, which means that the Riemann-Cartan curvature tensor is equal to zero: relation (2) is just the linear approximation of the condition of teleparallelism.

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Note that there exists few works of Mathematics about the incompatibility operator per-se, or about incompatible fields, but recent contributions can be found, see e.g., [**?**, 2, 12].

Nonetheless, the concept of dislocation line is related to another scale of matter description, namely the mesoscale, where it appears as the one-dimensional set of singularity for the elastic strain ε and stress field $\sigma = \mathbb{A}\varepsilon$, with $\mathbb{A} = 2\mu\mathbb{I}_4 + \lambda\mathbb{I}_2$, the elasticity tensor (where λ and μ are the Lamé coefficients). It is indeed well-known that these fields are not square integrable at this scale. Proving a Kröner identity at the mesoscale such as

$$\mathrm{inc}\varepsilon = \mathrm{Curl}\ \kappa_{\mathcal{L}},$$

was carried on by the author in a series of works [18–20] for some simple families of lines. Though, a proof of such relations for general lines was still missing. It is the purpose of this paper to propose a proof by studying pointwise and distributional properties of fields which posses a jump around dislocation lines, and are thus understood by means of functions of bounded variation.

Notations and conventions. Let $\mathbb{M}^3$ denote the space of square 3-matrices, and $\mathbb{S}^3$ that of symmetric 3-matrices. Let $E \in \mathbb{S}^3$ and $\beta \in \mathbb{M}^3$. We will sometimes use the following shortcut notation:

$$E = \mathbb{E}_f(\sigma) \Leftrightarrow -\mathrm{div}\sigma = f, \text{ almost everywhere in } \Omega, \text{ where } \sigma = \mathbb{A}E, \tag{3}$$

$$E = \mathbb{D}(\kappa_{\mathcal{L}}) \Leftrightarrow \mathrm{inc}E = \mathrm{Curl}\ \kappa_{\mathcal{L}}, \tag{4}$$

$$\beta = \mathbb{B}(\Lambda_{\mathcal{L}}) \Leftrightarrow \mathrm{Curl}\ \beta = \Lambda_{\mathcal{L}}^T. \tag{5}$$

The divergence and curl of a tensor E are defined componentwise as $(\mathrm{div}E)_i := \partial_j E_{ij}$ and $(\mathrm{Curl}\ E)_{ij} := \epsilon_{jkl}\partial_k E_{il}$, respectively. The incompatibility of a tensor E is the symmetric[1] tensor defined componentwise as follows[2]:

$$(\mathrm{inc}E)_{ij} := (\mathrm{Curl}\ \mathrm{Curl}^T\ E)_{ij} = \epsilon_{ikm}\epsilon_{jln}\partial_k\partial_l E_{mn}, \tag{6}$$

where subscript t stands for the transpose of a matrix.

The symmetric and skew-symmetric parts of a tensor M are denoted by M^S and M^A, respectively. Similarly, the symmetric and skew-symmetrci parts of a gradient ∇u are denoted by $\nabla^S u$ and $\nabla^A u$, respectively.

The functional space of (finite) vector-valued Radon measures, $\mathcal{M}(\Omega, \mathbb{R}^3)$, is defined as the dual space of $\mathcal{C}_c(\Omega, \mathbb{R}^3)$, that of tensor-valued Radon measures, $\mathcal{M}(\Omega, \mathbb{M}^3)$, as the dual space of $\mathcal{C}_c(\Omega, \mathbb{M}^3)$. A function u is said of bounded variation if $u \in L^1(\Omega)$ and if its distributional gradient Du is a Radon measure. Moreover, one writes

$$u \in SBV(\Omega)$$

to mean that u is of bounded variation and that Du is decomposed additively in two terms, the first which is absolutely continus w.r.t. Lebesgue measure on Ω, and the second which is concentrated on the jump set of u. Moreover,

$$|\Lambda|_{\mathcal{M}} := \sup_{\substack{\varphi \in \mathcal{C}_c(\Omega): \\ \|\varphi\|_\infty \le 1}} |\langle \Lambda, \varphi \rangle_{\mathcal{M}}|,$$

where $\langle \Lambda, \cdot \rangle_{\mathcal{M}}$ stands for the duality pairing. We refer to [1] for an introduction to the mathematical properties of these functions.

[1] Symmetry is intended with $(\mathrm{inc}E)$ seen as a distribution tenor.

[2] This definition is intended in Cartesian components, whereas in curvilinear basis, the componentwise definition must be adapted [21].

Aim of the work. The aim of this work is to prove existence and functional properties of a symmetric tensor E such that both conditions $E = \mathbb{E}_f(\sigma)$ and $E = \mathbb{D}(\kappa_{\mathcal{L}})$ hold true, provided the external volume force f and contortion tensor $\kappa_{\mathcal{L}}$ (i.e., dislocation density) are prescribed. Let us remark that, as proved in [14, 15], there exists a a unique solenoidal solution F to problem

$$\text{Curl } F = \mu,$$

where mu is a divergence-free measure. However, this result is not sufficient to deduce the relation $\text{inc}\varepsilon = \text{Curl}\,(\,\text{Curl}^T\ \varepsilon) = \text{Curl}\ \kappa_{\mathcal{L}}$, one reason being that $\kappa_{\mathcal{L}}^T$ is not in general a divergence-free measure (as opposed to $\Lambda_{\mathcal{L}}^T$, and except for pure edge dislocation loops), and another the fact that F is not in general a symmetric tensor.

For this reason another procedure must be undertaken to achieve our aim. The outline of the method is as follows: we first establish pointwise properties of the solution of the elasticity problem with jump on a surface enclosed by the dislocation. Then distributional properties of such solutions allow us to prove that there exist a β such that $\beta = \mathbb{B}(\Lambda_{\mathcal{L}})$, that is, we obtain the classical definition of a dislocation in finite elasticity, here holding true for infinitesimal strains. In the main theorem, we simply collect all preliminary result and compute the curl of $\text{Curl}^T\ \beta^S$ to get the sought result.

2. Preliminary results

The aim of this section is to prove that in the presence of a dislocation line in linear elasticity, there exists a strain E such that (4) and (5) are satisfied. To this aim, a series of results about fields of bounded variation and deformation must be proved.

Lemma 1. *Let $\mathcal{L}$ be a Lipschitz closed curve in $\mathbb{R}^3$ and S a bounded Lipschitz surface with boundary $\mathcal{L}$ and unit normal N. Let $B \in \mathbb{R}^3$. The solution of*

$$\begin{cases} \text{div}(\mathbb{A}\nabla w) = 0 & in \quad \mathbb{R}^3 \setminus S \\ [\![w]\!] := w^+ - w^- = B & on \quad S \\ [\![(\mathbb{A}\nabla w)N]\!] := ((\mathbb{A}\nabla w)N)^+ - ((\mathbb{A}\nabla w)N)^- = 0 & on \quad S \end{cases} \tag{7}$$

is given componentwise by

$$w_i(x) = -B_j \int_S (\mathbb{A}\nabla\Gamma(x' - x\cdot)N(x'))_{ij}\, d\mathcal{H}^2(x'), \tag{8}$$

for $x \in \mathbb{R}^3 \setminus S$, where Γ is the solution in $\mathbb{R}^3$ of $\text{div}(\mathbb{A}\nabla\Gamma) = \delta_0\mathbb{I}_2$.

Proof. Let $S \subset \hat{\Omega}$ be a smooth surface of discontinuity bounded by $\mathcal{L}$. Let $S^- \neq S$ be another smooth surface bounded by $\mathcal{L}$ and staying below S. Let V be the volume comprised between S and S^- and $S_V := S \cup S^-$ with outer unit normal N be such that $\partial V := S_V$. Supposing that w is smooth enough and summable in $\mathbb{R}^3$, we have the following identities in V:

$$\int_V \partial'_k(\partial'_l w_j(x')\Gamma_{ip}(x' - x))dx' = \int_{S_V} \partial'_l w_j(x')\Gamma_{ip}(x' - x)N_k(x')d\mathcal{H}^2(x')$$

and

$$\int_V \partial'_l(w_j(x')\partial'_k\Gamma_{ip}(x' - x))dx' = \int_{S_V} w_j(x')\partial'_k\Gamma_{ip}(x' - x)N_l(x')d\mathcal{H}^2(x').$$

Thus by subtraction it holds

$$\int_V \partial'_k\partial'_l w_j(x')\Gamma_{ip}(x' - x))dx' - \int_V w_j(x')\partial'_k\partial'_l\Gamma_{ip}(x' - x))dx'$$
$$= \int_{S_V} (\partial'_l w_j(x'))^- \Gamma_{ip}(x' - x)N_k(x')d\mathcal{H}^2(x') - \int_{S_V} w_j^-(x')\partial'_k\Gamma_{ip}(x' - x)N_l(x')d\mathcal{H}^2(x').$$

Moreover, the same identities in $\mathbb{R}^3 \setminus \bar{V}$ yield

$$\begin{aligned}&\int_{\mathbb{R}^3\setminus\bar{V}} \partial'_k\partial'_l w_j(x')\Gamma_{ip}(x'-x))dx' - \int_{\mathbb{R}^3\setminus\bar{V}} w_j(x')\partial'_k\partial'_l\Gamma_{ip}(x'-x))dx' \\ = & -\int_{S_V} (\partial'_l w_j(x'))^+\,\Gamma_{ip}(x'-x)N_k(x')d\mathcal{H}^2(x') + \int_{S_V} w_j^+(x')\partial'_k\Gamma_{ip}(x'-x)N_l(x')d\mathcal{H}^2(x'),\end{aligned}$$

and hence, by summing,

$$\begin{aligned}&\int_{\mathbb{R}^3\setminus S_V} \partial'_k\partial'_l w_j(x')\Gamma_{ip}(x'-x))dx' - \int_{\mathbb{R}^3\setminus S_V} w_j(x')\partial'_k\partial'_l\Gamma_{ip}(x'-x))dx' \\ = & -\int_{S_V} [\![\partial'_l w_j(x')]\!]\Gamma_{ip}(x'-x)N_k(x')d\mathcal{H}^2(x') + \int_{S_V} [\![w_j(x')]\!]\partial'_k\Gamma_{ip}(x'-x)N_l(x')d\mathcal{H}^2(x').\end{aligned}$$

Contracting with $\mathbb{A}_{ljki}$ yields

$$\begin{aligned}&\int_{\mathbb{R}^3\setminus S_V} (\operatorname{div}(\mathbb{A}\nabla w)_p(x')\Gamma(x'-x))dx' - \int_{\mathbb{R}^3\setminus S_V} w_j(x')(\operatorname{div}(\mathbb{A}\nabla\Gamma)_{jp}(x'-x))dx' \\ = & -\int_{S_V} [\![\mathbb{A}\nabla' w(x')N]\!]_i\Gamma_{ip}(x'-x)d\mathcal{H}^2(x') + \int_{S_V} [\![w_j(x')]\!]\left(\mathbb{A}\nabla'\Gamma(x'-x)N\right)_{jp} d\mathcal{H}^2(x'),\end{aligned} \tag{9}$$

that is, for $x \in \mathbb{R}^3 \setminus S_V$,

$$\begin{aligned}&\int_{\mathbb{R}^3\setminus S_V} (\operatorname{div}(\mathbb{A}\nabla w)_i(x')\Gamma_{ip}(x'-x))dx' - w_p(x) \\ = & -\int_{S_V} [\![\mathbb{A}\nabla' w(x')N]\!]_i\Gamma_{ip}(x'-x)d\mathcal{H}^2(x') \\ + & \int_{S_V} [\![w_j(x')]\!]\left(\mathbb{A}\nabla'\Gamma(x'-x)N\right)_{jp} d\mathcal{H}^2(x').\end{aligned} \tag{10}$$

Taking the particular[3]

$$w = \int_S \left(\mathbb{A}\nabla\Gamma(y-\cdot)\right)N(y)Bd\mathcal{H}^2(y),$$

w satisfies $\operatorname{div}(\mathbb{A}\nabla w)(x) = 0$ for $x \in \mathbb{R}^3 \setminus S$, and hence, by (10) and for $x \in \mathbb{R}^3 \setminus S_V$,

$$\begin{aligned}w_p(x) = & \int_{S_V} [\![\mathbb{A}\nabla' w(x')N]\!]_i\Gamma_{ip}(x'-x)d\mathcal{H}^2(x') \\ - & \int_{S_V} [\![w_j(x')]\!]\left(\mathbb{A}\nabla'\Gamma(x'-x)N\right)_{jp} d\mathcal{H}^2(x').\end{aligned}$$

Keeping this w, consider now any smooth tensor test fuction φ with compact support in place of the tensor Γ. By (10), it holds

$$\begin{aligned}&\int_{\mathbb{R}^3\setminus S_V} w_j(x')(\operatorname{div}(\mathbb{A}\nabla\varphi)_{jp}(x'))dx' = \int_{\mathbb{R}^3} w_j(x')(\operatorname{div}(\mathbb{A}\nabla\varphi)_{jp}(x'))dx' \\ = & \int_{S_V} [\![\mathbb{A}\nabla' w(x')N]\!]_i\varphi_{ip}(x')d\mathcal{H}^2(x') - \int_{S_V} [\![w_j(x')]\!]\left(\mathbb{A}\nabla'\varphi(x')N\right)_{jp} d\mathcal{H}^2(x').\end{aligned} \tag{11}$$

Define the distribution γ_B concentrated in S as

$$\langle\gamma_B, \varphi\rangle := -\int_S (\mathbb{A}\nabla\varphi)N(y)Bd\mathcal{H}^2(y).$$

[3]Here we omit the intermediate step of taking a cut-off function.

By definition, $w(x) = \int_S (\mathbb{A}\nabla\Gamma)(x-y)N(y)Bd\mathcal{H}^2(y) = -\langle\gamma_B, \Gamma(x-\cdot)\rangle$. Observe that

$$\operatorname{div}(\mathbb{A}\nabla w) = -\gamma_B \tag{12}$$

holds in the distribution sense, since for any smooth test function with compact support φ, by definition of the convolution between distributions [16], one has

$$\begin{aligned}\langle \operatorname{div}(\mathbb{A}\nabla w), \varphi\rangle = \langle w, \operatorname{div}(\mathbb{A}\nabla\varphi)\rangle &= -\langle\langle\gamma_B, \Gamma(x-\cdot)\rangle, \operatorname{div}(\mathbb{A}\nabla\varphi)(x)\rangle \\ &= -\langle\gamma_B, \langle \operatorname{div}(\mathbb{A}\nabla\Gamma)(x-\cdot), \varphi(x)\rangle\rangle \\ &= -\langle\gamma_B, \varphi\rangle. \end{aligned}\tag{13}$$

Substracting (13) from (11) yields

$$\begin{aligned} 0 &= \int_{S_V} [\![\mathbb{A}\nabla' w(x')N]\!]_i \varphi_{ip}(x')d\mathcal{H}^2(x') - \int_S [\![w_j(x') - B_j]\!] \left(\mathbb{A}\nabla'\varphi(x')N\right)_{jp} d\mathcal{H}^2(x') \\ &- \int_{S^-} [\![w_j(x')]\!] \left(\mathbb{A}\nabla'\varphi(x')N\right)_{jp} d\mathcal{H}^2(x'), \end{aligned}\tag{14}$$

which since it holds for any test function φ, yields (7) by (12), achieving the proof. □

Remark that taking an arbitrary $\partial_N\varphi$ on S^- while $\partial_N\varphi = \varphi = 0$ on S in (14) yields the continuity of w on S^-. Moreover, by (8), it holds

$$\partial_k w_i(x) = -B_j \int_S \partial_k \left(\mathbb{A}\nabla\Gamma(y-x)N(y)\right)_{ij} d\mathcal{H}^2(y). \tag{15}$$

More results on this topic can be found in [4].

Lemma 2. *Let $\mathcal{L} \subset \Omega$ be the union of a finite number of smooth dislocation loops and $S \subset \Omega$ a smooth surface enclosed by $\mathcal{L}$. Referring to Lemma 1, let w be the solution of*

$$-\operatorname{div}(\mathbb{A}\nabla w) = 0 \quad in \quad \mathbb{R}^3 \setminus S, \quad [\![w]\!] = B, \quad [\![(\mathbb{A}\nabla w)N]\!] = 0 \quad on \quad S.$$

Then $w \in SBV(\Omega, \mathbb{R}^3)$, $\nabla w \in L^p(\Omega, \mathbb{R}^3)$ for $1 \le p < 2$ and

$$-\operatorname{Curl}\,\bar{\nabla} w = \Lambda_{\mathcal{L}}^T,$$

in the distribution sense, where $\overline{\nabla} w$ is the absolutely continuous part of the distributional derivative Dw in Ω (that is, $\nabla w = \bar{\nabla} w$ almost everywhere). Moreover $-\operatorname{div}(\mathbb{A}\nabla w) = 0$ in $\mathbb{R}^3 \setminus \mathcal{L}$, $w \in \mathcal{C}^\infty(\Omega \setminus \mathcal{L}, \mathbb{R}^3)$ and it holds

$$|\nabla w(x)| \le c|B||\mathcal{L}|(1 + \frac{1}{d(x, \mathcal{L})}), \tag{16}$$

with c a constant depending on the line curvature, and $|\mathcal{L}|$ its length.

Proof. The second part of the statement, namely (16), is proven as in Lemma 4 of [15] by estimating $|\partial_i w(x)|$ by means of formula (15), and up to a positive factor given by the uniform bound of $\mathbb{A}$. Let us now prove the first part of the statement in the case of a smooth $\mathcal{L}$. Let w be a solution to (7). By (16), $\nabla w \in L^p(\Omega, \mathbb{M}^3)$ for $p < 2$. It has been shown that w is smooth outside S where it has a jump of amplitude $b := |B|$. In particular this means that w belongs to $SBV(\Omega, \mathbb{R}^3)$ with its distributional derivative given by

$$\langle Dw, \varphi\rangle := -\langle w, \operatorname{div}\varphi\rangle = S(\varphi) + \langle\nabla w, \varphi\rangle, \tag{17}$$

for all $\varphi \in \mathcal{D}(\Omega, \mathbb{M}^3)$, and where S denotes the distribution $S(\varphi) = -\int_S N_j B_i \varphi_{ij} d\mathcal{H}^2$.

Let us prove that $-\operatorname{Curl}\,\nabla w = \Lambda_{\mathcal{L}}^T$. To this aim let us take $\psi \in \mathcal{D}(\Omega, \mathbb{M}^3)$ and write

$$\begin{aligned} -\langle \operatorname{Curl}\,\nabla w, \psi\rangle := -\langle\nabla w, \operatorname{Curl}\,\psi\rangle &= -\langle Dw, \operatorname{Curl}\,\psi\rangle + S(\operatorname{Curl}\,\psi) \\ &= \int_C \tau_j B_i \psi_{ij} d\mathcal{H}^1, \end{aligned}\tag{18}$$

where the second equality follows from (17) with $\varphi = \operatorname{Curl}\,\psi$, and the third one by Stokes theorem and the distributional identity $\operatorname{Curl}\,\operatorname{Div} = 0$.

In order to prove that $\text{Div}\ \nabla w = 0$, let $\hat{S} \supset S$ such that $\hat{S}$ separates Ω in two parts Ω^- and Ω^+. Then for every test function $\varphi \in C_c^\infty(\Omega, \mathbb{R}^3)$ it holds

$$\int_\Omega \nabla w \nabla \varphi dx = \int_{\Omega^+} \nabla w \nabla \varphi dx + \int_{\Omega^-} \nabla w \nabla \varphi dx =$$
$$- \int_{\Omega^+} \text{Div}\ \nabla w \varphi dx - \int_{\Omega^-} \text{Div}\ \nabla w \varphi dx + \int_{\hat{S}^+} \partial_N w^+ \varphi dx - \int_{\hat{S}^-} \partial_N w^- \varphi dx = 0,$$

achieving the proof.

□

Remark 1. *Since $w \in L^1(\Omega)$ and $\nabla w \in L^p(\Omega)$ with $1 \le p < 2$, Sobolev's embedding yields $w \in L^{p*}(\Omega)$, and hence $w \in W^{1,p}(\Omega)$ and the trace of w, $w_{|\partial\Omega} \in W^{1/p',p}(\partial\Omega)$. On the other hand* (16) *implies that $\nabla w \in L^p(\partial\Omega)$ with $1 \le p < 2$.*

3. Main result: Kröner relation

Let $\varphi \in W_0^{1,p'}(\Omega, \mathbb{R}^3)$ with $p' > 3$. Then by Sobolev embedding, $\varphi \in \mathcal{C}^0(\Omega, \mathbb{M}^3)$, and $|\langle \Lambda_\mathcal{L}, \varphi \rangle| \le C\|\varphi\|_\infty \le C'\|\varphi\|_{W^{1,p'}}$ and hence

$$\Lambda_\mathcal{L}, \kappa_\mathcal{L} \in W^{-1,p}(\Omega), \quad 1 \le p < 3/2. \tag{19}$$

In the following theorem we prove Kröner relation at the mesoscopic scale, i.e., $\text{inc}\varepsilon = \text{Curl}\ \kappa_\mathcal{L}$. The condition $\varepsilon \in L^p(\Omega)$, with $1 \le p < 2$ immediately entails that $\text{inc}\varepsilon \in W^{-2,p}(\Omega)$, since $\Phi \in W_0^{2,p'}(\Omega, \mathbb{M}^3)$ with $p' > 2$ yields

$$|\langle\, \text{inc}\varepsilon, \Phi\rangle| = |\langle \varepsilon,\, \text{inc}\Phi\rangle| \le C\|\varepsilon\|_{L^p(\Omega)}\|\Phi\|_{W^{2,p'}(\Omega)}.$$

Note however, that by virtue of (19), Kröner relation must be understood as $\text{inc}^*\varepsilon = \text{Curl}\ \kappa_\mathcal{L}$, where $\text{inc}^*\varepsilon$ is the restriction of $\text{inc}\varepsilon$ to $W_0^{2,p'}(\Omega)$ for $p' > 3$.

Theorem 1. *Let $f \in \mathcal{C}^\infty(\Omega)$ and $g \in \mathcal{C}^\infty(\partial\Omega)$. Under the hypotheses of Lemma 2, there exists $u \in SBV(\Omega, \mathbb{R}^3)$ such that $\bar{\nabla} u \in L^p(\Omega, \mathbb{M}^3)$ for $1 \le p < 2$ and satisfying $\bar{\nabla}^S u = \mathbb{E}_f(\sigma) \in L^p(\Omega, \mathbb{S}^3)$ and $\bar{\nabla} u = \mathbb{B}(\Lambda_\mathcal{L})$. Furthermore, $\bar{\nabla}^S u = \mathbb{D}(\kappa_\mathcal{L})$, where $\kappa_\mathcal{L} \in W^{-1,p}(\Omega)$ with $1 \le p < 3/2$.*

Proof. Let w be the vector of Lemma 2. Then

$$-\,\text{Curl}\ \bar{\nabla} w = \Lambda_\mathcal{L}^T.$$

Let $v \in H^1(\Omega, \mathbb{R}^3)$ be a weak solution to[4]

$$-\,\text{div}(\mathbb{A}\nabla^S v) = -f \quad \text{in} \quad \Omega, \quad (\mathbb{A}\nabla^S v)N = -g - (\mathbb{A}\nabla w)N \quad \text{on} \quad \partial\Omega \setminus S,$$

with the value of $(\mathbb{A}\nabla w)N$ on $\partial\Omega$ provided by (15). Then, $u := -(v + w)$ satisfies

$$-\,\text{div}(\mathbb{A}\bar{\nabla}^S u) = f \quad \text{in} \quad \Omega \setminus S, \quad (\mathbb{A}\bar{\nabla}^S u)N = g \quad \text{on} \quad \partial\Omega \setminus S. \tag{20}$$

Remark that if instead, one poses $v = -w_{|\partial\Omega}$ on $\partial\Omega$, then[5] $u = 0$ on $\partial\Omega$.

In principle this solution u depends on the choice of surface S. However, taking another surface S' enclosed by $\mathcal{L}$ will produce a shift of u by $-B$ in the volume bounded by $S \cup S'$, and hence will not affect the absolutely continuous part of its distributional gradient $\bar{\nabla} u$. Moreover, w will be smooth in $\Omega \setminus S$ for any choice of S, from which it is deduced that w is smooth in $\Omega \setminus \mathcal{L}$. Therefore $\bar{\nabla} u$ is independent of S.

Since $[\![u]\!] = -B$ on S and $[\![(A\bar{\nabla} u)N]\!] = 0$ on S, one has

$$\text{Curl}\ \bar{\nabla} u = -\,\text{Curl}\ \bar{\nabla} w = \Lambda_\mathcal{L}^T,$$

[4] By Remark 1, one has $(\mathbb{A}\nabla^S w)N \in L^{4/3}(\partial\Omega)$ and hence a solution exists in $H^1(\Omega)$ by [3, Theorem 6.3.5].

[5] A weak solution also exists in this case, since by Remark 1, one has $w_{|\partial\Omega} \in W^{1/p',p}(\partial\Omega, \mathbb{R}^3)$ for $1 \le p < 2$. Now, by classical lifting theorems, the non-homogeneous problem is recast into a homogeneous problem with a right-hand side in $W^{-1,p}(\Omega, \mathbb{R}^3)$ for which a solution $v \in W^{1,p}(\Omega, \mathbb{R}^3)$ exists as shown in [20].

with $\bar\nabla u \in L^1(\Omega)$ by virtue of Lemma 2, and recalling that $\nabla v = Dv$ is intended in the distribution sense, and $[\![v]\!] = 0$. Moreover, $\bar\nabla^S u \in L^p(\Omega)$. Now, by identity $\bar\nabla u = \bar\nabla^S u + \bar\nabla^A u$, one has

$$\text{Curl } \bar\nabla^S u = \Lambda_{\mathcal{L}}^T - \text{ Curl } \bar\nabla^A u = \Lambda_{\mathcal{L}}^T - \nabla^T\omega + \mathbb{I}_2 \,\text{div}\omega,$$

where one has componentwise $(\bar\nabla^A u)_{ij} = \epsilon_{ijk}\omega_k$ and $\omega_i = \frac{1}{2}\epsilon_{ikl}(\bar\nabla u)_{kl}$. Furthermore, $\text{div}\omega = \partial_i\omega_i = -\frac{1}{2}\,\text{tr}\,\text{Curl } \bar\nabla u = -\frac{1}{2}\,\text{tr}\Lambda_{\mathcal{L}}$, and hence

$$\text{Curl } \bar\nabla^S \bar u = \kappa_{\mathcal{L}}^T - \nabla^T\omega. \tag{21}$$

Now, $\omega \in L^1(\Omega, \mathbb{R}^3)$ and hence $\nabla\omega = D\omega$ is intended in the distribution sense, in such a way that Curl $\nabla\omega = 0$, yielding

$$\text{inc}\bar\nabla^S u = \text{ Curl } \text{ Curl }^T\bar\nabla^S u = \text{ Curl } (\kappa_{\mathcal{L}} - D\omega) = \text{ Curl } \kappa_{\mathcal{L}},$$

achieving the proof. □

4. Concluding remarks

Kröner relation is often mentioned in the literature but a complete proof was missing at the mesoscale. By means of this formula, it was the aim of this paper to make the link between functions of bounded variation, viz., the displacement field u, and dislocations at the mesoscopic scale. This formula shows several important features. First, the role of the contortion, in place, or in parallel, of the dislocation density. It turns out that the contortion has a clear geometrical meaning related to the metric torsion in the presence of dislocations [9, 17]. Second, it shows the crucial role of the incompatibility operator. Indeed, this operator is related to the Beltrami decomposition of symmetric tensors, namely $\varepsilon = \nabla^S u + \text{inc}F$ (see, e.g., [10]), where inc$F$ is the part of the elastic strain, which is incompatible. Note that once such a relation is proved, the strain satisfies

$$\text{inc}\varepsilon = \text{ inc}\,\text{inc}F = \text{ Curl } \kappa,$$

putting light on a special 4th-order operator, inc inc, whose mathematical properties, among which coercivity (that is, ellipticity) were studied in [2].

Lastly, this formula teaches us that under the assumption of linearized elasticity, where the skewsymmetric part of $\bar\nabla u$ (recall that $\bar\nabla u$ stands for the absolutely continuous part of Du w.r.t. Lebesgue measure in Ω) is not taken into account, the relation between deformation and dislocation density might be given by the incompatibility of $\bar\nabla^S u$, precisely by Kröner's formula, in place of the classical Curl $\bar\nabla u = \Lambda_{\mathcal{L}}^T$, valid for finite as well as for infinitesimal elastic strains, which would require to also consider the skewsymmetric part, for which no Poincaré-Korn-types of bounds do exist.

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