EXISTENCE FOR A ONE-EQUATION TURBULENT MODEL WITH STRONG NONLINEARITIES

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Abstract. The purpose of this article is to improve the existence theory for the steady problem of an one-equation turbulent model. For this study, we consider a very general model that encompasses distinct situations of turbulent flows described by the k-epsilon model. Although the boundary-value problem we consider here is motivated by the modelling of turbulent flows through porous media, the importance of our results goes beyond this application. In particular, our results are suited for any turbulent flows described by the k-epsilon model whose mean flow equation incorporates a feedback term, as the Coriolis force, the Lorentz force or the Darcy-Forchheimer's drag force. The consideration of feedback forces in the mean flow equation will affect the equation for the turbulent kinetic energy (TKE) with a new term that is known as the production and represents the rate at which TKE is transferred from the mean flow to the turbulence. For the associated boundary-value problem, we prove the existence of weak solutions by assuming that the feedback force and the turbulent dissipation are strong nonlinearities, *i.e.* when no upper restrictions on the growth of these functions with respect to the mean velocity and to the turbulent kinetic energy, respectively, are required. This result improves, in particular, the existence theory for the classical turbulent k-epsilon model which corresponds to assume that both the feedback force and the production term are absent in our model.

Key words and phrases: turbulence, $k-$epsilon modelling, strong nonlinearities, existence.

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1. Introduction

In the literature of Fluid Mechanics, there are two distinct approaches to describe turbulent flows through porous media by using the k-epsilon model. For the first approach, the turbulent transport equations are derived by volume averaging the Reynolds-averaged microscopic equations (see *e.g.* [16, 21]). As for the second approach, the turbulent transport equations are derived by time averaging the extended Darcy-Forchheimer model obtained by volume-averaging the microscopic equations (see *e.g.* [1]). Both techniques aim to derive suitable macroscopic transport equations, but, as the authors of [1] have concluded, turbulent models derived directly from the general macroscopic equations do not accurately characterize turbulence induced by the porous medium. The difference on the order of application of the two concepts of average gives rise to distinct equations for the transport of the turbulent kinetic energy. This still happens even if we apply the average concepts by the same order but use different techniques to model the production term (compare [21] with [16]). Motivated essentially by the models [16, 21], we consider, in this work, the following general boundary-value problem,

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = \mathbf{g} - \mathbf{f}(\mathbf{u}) - \boldsymbol{\nabla} p + \mathbf{div}\left((\nu + \nu_T(k))\,\mathbf{D}(\mathbf{u})\right) \quad \text{in } \Omega, \tag{1.2}$$

$$\mathbf{u}\cdot\boldsymbol{\nabla} k = \operatorname{div}\left(\nu_D(k)\boldsymbol{\nabla} k\right) + \nu_T(k)|\mathbf{D}(\mathbf{u})|^2 + P(\mathbf{u},k) - \varepsilon(k) \quad \text{in } \Omega, \tag{1.3}$$

$$\mathbf{u} = \mathbf{0} \quad \text{and} \quad k = 0 \quad \text{on } \partial\Omega, \tag{1.4}$$

where Ω is a bounded domain of $\mathbb{R}^d$, $d \geq 2$, with its boundary denoted by $\partial\Omega$.

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In the scope of turbulent flows through porous media, the porous domain Ω is the so-called matrix or frame and it is assumed to be rigid, fixed, isotropic and saturated by an incompressible fluid. The velocity vector field $\mathbf{u}$, the pressure p and the external forces field $\mathbf{g}$ are, in fact, averages that result by the application of double averaging concepts (see *e.g.* [13]). The averaged tensor $\mathbf{D}(\mathbf{u})$ is the symmetric part of the averaged gradient $\boldsymbol{\nabla}\,\mathbf{u}$, the positive constant ν is the kinematic viscosity and expresses the ratio of the internal forces in the fluid, called dynamic viscosity, to the mass density ρ, assumed to be constant and positive. It should be noted that (in the scope of porous media) all the terms in the momentum equation (1.2) should come affected by the porosity of the medium, say ϕ, which is obtained by spatial averaging the characteristic function of the fluid phase, and therefore may depend on the space variable, and ranging in the interval $(0, 1)$ (see *e.g.* [13]). Similarly to what we assumed that other properties of the fluid are constant, such as kinematic viscosity and density, in this work we assume the porosity is also constant. The feedback term $\mathbf{f}(\mathbf{u})$, that describes the resistance made by the rigid matrix of the porous medium to the flow, is characterized by the Darcy-Forchheimer law,

$$\mathbf{F}(\mathbf{u}) = c_{Da}\mathbf{u} + c_F\,|\mathbf{u}|\mathbf{u}, \tag{1.5}$$

where c_{Da} and c_F are the Darcy and Forchheimer coefficients, positive constants that are experimentally determined. The scalar function k is an unknown of the problem that characterizes the energy of turbulence in the flow, and therefore it is usually called turbulent kinetic energy (TKE). The rate of dissipation of the TKE is described, in the model, by the function ε which is denoted by dissipation of the TKE, or, briefly, turbulent dissipation. The scalar function ν_T is the (Boussinesq) turbulent viscosity, or eddy viscosity, that, according to Prandtl's hypothesis (see *e.g.* [7]), may depend on k and on ε, whereas ν_D is the turbulent diffusion and may also depend on k and on ε (see *e.g.* [15]). The emergence of the quantity ε in the model, would led us to derive an equation for the transport of this function in order to close the model. However, the consideration of one-equation models, that we assume in this work, is acceptable in the sense that the equation for ε may be discarded by prescribing an appropriate length scale. Consequently the turbulent viscosity ν_T and the turbulent diffusion ν_D are assumed to depend only on k, and, due to Prandtl's hypothesis, the turbulent dissipation ε depends only on k, being considered, in most studies, the Launder-Spalding hypothesis, *i.e.* that ε is of the order of $k^{\frac{3}{2}}$. The additional term $P(\mathbf{u}, k)$ in equation (1.3), that appears as an output of the averaging process, is a production term of turbulent kinetic energy that gives account of the solids inside the fluid and is distinct for each model [16, 21]. In fact, we have

$$P(\mathbf{u}, k) \equiv P(\mathbf{u}) := C_{NK}|\mathbf{u}|^3, \quad \text{for the model [16]} \tag{1.6}$$

$$P(\mathbf{u}, k) := C_{PL}|\mathbf{u}|\,k, \quad \text{for the model [21]}, \tag{1.7}$$

where C_{NK} and C_{PL} are positive constants that are experimentally determined in each model.

Problem (1.1)-(1.4) is very general and it encompasses other situations of turbulence modelling not directly related with porous media. In fact, this problem can be used to model turbulent flows in a rotating frame, where the feedback term $\mathbf{f}(\mathbf{u})$ accounts for the Coriolis acceleration and, in that case, the production term $P(\mathbf{u}, k)$ is zero. The problem (1.1)-(1.4) may also be adapted to study turbulent flows controlled by a given magnetic field, where, in that case, the feedback term $\mathbf{f}(\mathbf{u})$ accounts for the Lorentz force, a term where the Navier-Stokes equations are coupled to Maxwell's equations and Ohm's law (see [20] and the references cited therein). In particular, by taking $\mathbf{f}(\mathbf{u}) = \mathbf{0}$, $P(\mathbf{u}, k) = 0$ and assuming the turbulent dissipation $\varepsilon(k)$ is of the order of $k^{\frac{3}{2}}$, we recover the steady version of the one-equation turbulent k–epsilon model (see *e.g.* [7, 15]). One-equation problems of the turbulent k–epsilon model have been investigated during the last 20 years, although important questions, as the 3-d transient problem, or the case of real turbulent viscosity and turbulent diffusion functions, remain open. For questions of existence, uniqueness and regularity of the solutions, related to the problem (1.1)-(1.4) with $\mathbf{f}(\mathbf{u}) = \mathbf{0}$ and $P(\mathbf{u}, k) = 0$, we address the reader to the works [9, 10, 11, 12, 14, 18].

The mathematical analysis of the problem (1.1)-(1.4) has started, to our best knowledge, in the work [19], and in [20] it was proved the existence of weak solutions under suitable growth conditions on the feedback terms $\mathbf{f}(\mathbf{u})$, $\varepsilon(k)$ and $P(\mathbf{u}, k)$. In [20] we also have proved a uniqueness result for this problem under usual smallness conditions on the problem data, but with restrictive conditions on the integrability of the gradient solutions.

This article is organized as follows. In the current Section 1 we made the introduction to our work and in Section 2 we define the notion of solutions to the problem (1.1)-(1.4) we are interested in. The purposes of this paper are addressed still in Section 2, but the main result, Theorem 3.1, is presented in Section 3. Great part of the paper is dedicated to prove Theorem 3.1 whose proof starts in Section 3 and is carried out then in Sections 4 and 5.

2. Weak formulation of the problem

In order to define the notion of a weak solution to the problem (1.1)-(1.4), let us introduce the following function spaces largely used in the mathematical analysis of fluid problems,

$$\begin{aligned}
&\mathcal{V} := \{\mathbf{v} \in \mathbf{C}_0^\infty(\Omega) : \operatorname{div}\mathbf{v} = 0\},\\
&\mathbf{H} := \text{closure of } \mathcal{V} \text{ in } \mathbf{L}^2(\Omega),\\
&\mathbf{V} := \text{closure of } \mathcal{V} \text{ in } \mathbf{H}^1(\Omega).
\end{aligned}$$

By $\mathbf{V}'$ we shall denote the dual of the space $\mathbf{V}$. In the mathematical treatment of the turbulent problem (1.1)-(1.4), there is a set of usual assumptions that although do not follow from the real situation they are physically admissible,

$$\mathbf{f} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d \quad \text{is a Carathéodory function,} \tag{2.1}$$

$$\varepsilon,\ \nu_T,\ \nu_D : \Omega \times \mathbb{R} \to \mathbb{R} \quad \text{are Carathéodory functions,} \tag{2.2}$$

$$P : \Omega \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \quad \text{is a Carathéodory function.} \tag{2.3}$$

Observe that, in view of these assumptions, we are considering the possibility of some, or even all, of these functions might depend on the space variables. In particular, assumption (2.2) fits with turbulent dissipation, turbulent viscosity and turbulent diffusion functions involved in realistic models (see *e.g.* [7, 15, 20]).

There is another set of assumptions that impose some restrictions on the physics of the problem, but are mathematically needed. We assume the boundedness of both turbulent viscosity and turbulent diffusion,

$$|\nu_T(\mathbf{x}, k)| \le C_T, \quad |\nu_D(\mathbf{x}, k)| \le C_D \quad \text{for all } k \in \mathbb{R} \text{ and a.a. } \mathbf{x} \in \Omega, \tag{2.4}$$

for some positive constants C_T and C_D.

We are now in conditions to present a notion of weak solution to the problem (1.1)-(1.4) we are interested in this work.

Definition 2.1. *Let the conditions (2.1)-(2.3) and (2.4) be fulfilled and assume that* $\mathbf{g} \in \mathbf{V}'$. *We say a pair* $(\mathbf{u}, k)$ *is a weak solution to the problem (1.1)-(1.4), if:*

(1) $\mathbf{u} \in \mathbf{V}$ *and for every* $\mathbf{v} \in \mathbf{V} \cap \mathbf{L}^\infty(\Omega)$ *there hold* $\mathbf{f}(\mathbf{x}, \mathbf{u}) \cdot \mathbf{v} \in \mathbf{L}^1(\Omega)$ *and*

$$\int_\Omega (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} \cdot \mathbf{v}\, d\mathbf{x} + \int_\Omega (\nu + \nu_T(\mathbf{x}, k))\, \mathbf{D}(\mathbf{u}) : \boldsymbol{\nabla}\mathbf{v}\, d\mathbf{x} + \int_\Omega \mathbf{f}(\mathbf{x}, \mathbf{u}) \cdot \mathbf{v}\, d\mathbf{x} = \int_\Omega \mathbf{g} \cdot \mathbf{v}\, d\mathbf{x}; \tag{2.5}$$

(2) $k \in \mathrm{W}_0^{1,q}(\Omega)$, *with* $\frac{2d}{d+2} \le q < d'$, *and for every* $\varphi \in \mathrm{W}_0^{1,q'}(\Omega)$ *there hold* $\varepsilon(\mathbf{x}, k)\, \varphi,\ P(\mathbf{x}, \mathbf{u}, k)\, \varphi \in \mathrm{L}^1(\Omega)$ *and*

$$\int_\Omega (\mathbf{u} \cdot \boldsymbol{\nabla} k)\varphi\, d\mathbf{x} + \int_\Omega \nu_D(\mathbf{x}, k)\boldsymbol{\nabla} k \cdot \boldsymbol{\nabla}\varphi\, d\mathbf{x} + \int_\Omega \varepsilon(\mathbf{x}, k)\, \varphi\, d\mathbf{x} = \int_\Omega \nu_T(\mathbf{x}, k)|\mathbf{D}(\mathbf{u})|^2 \varphi\, d\mathbf{x} + \int_\Omega P(\mathbf{x}, \mathbf{u}, k)\, \varphi\, d\mathbf{x}; \tag{2.6}$$

(3) $k \ge 0$ *and* $\varepsilon(\mathbf{x}, k) \ge 0$ a.e. *in* Ω.

Remark 2.1. *Note that condition* $\frac{2d}{d+2} \le q < d'$ *in (2) of the above definition, is needed to control the first integral term of (2.6). In fact, to hold the boundedness of this term is needed that* $\mathrm{W}_0^{1,q}(\Omega) \hookrightarrow \mathrm{L}^2(\Omega)$ *and* $\mathrm{W}^{1,q'}(\Omega) \hookrightarrow \mathrm{L}^\infty(\Omega)$, *which in turn are valid, respectively for* $q \ge \frac{2d}{d+2}$ *and* $q < d'$, *due to Sobolev imbeddings. In particular, the need of* $q < d'$ *is of the utmost importance to control the fourth integral term of (2.6) when* $\mathbf{u}$ *merely belongs to* $\mathbf{V}$.

Remark 2.2. *The need of the test function* $\mathbf{v}$ *of (2.6) to be in* $\mathbf{L}^\infty(\Omega)$ *is to assure that* $\mathbf{f}(\mathbf{x}, \mathbf{u}) \cdot \mathbf{v} \in \mathbf{L}^1(\Omega)$ *if we solely have* $\mathbf{f}(\mathbf{x}, \mathbf{u}) \in \mathbf{L}^1(\Omega)$. *The same happens for the test function* φ *of (2.6) in order to have* $\varepsilon(\mathbf{x}, k)\, \varphi \in \mathrm{L}^1(\Omega)$ *if solely* $\varepsilon(\mathbf{x}, k) \in \mathrm{L}^1(\Omega)$. *But, in this case, it is enough to have* $\varphi \in \mathrm{W}_0^{1,q'}(\Omega)$, *because, due to the range of the considered* q *and as observed in the precedent remark, we have* $\mathrm{W}^{1,q'}(\Omega) \hookrightarrow \mathrm{L}^\infty(\Omega)$. *The application of this remark will be clear later on at Sections 5.2 and 5.3.*

To simplify the exposition, we assume, in the rest of the work, that the general space dimension d satisfies to

$$2 \le d \le 4.$$

In [20, Theorem 3.1] we have proved the existence of weak solutions to the problem (1.1)-(1.4) under the following growth conditions on the feedback forces field and on the turbulent dissipation function,

$$|\mathbf{f}(\mathbf{x},\mathbf{u})| \leq C_f|\mathbf{u}|^{\alpha} \quad \text{for} \quad 0 \leq \alpha \leq \frac{d+2}{d-2} \text{ if } d \neq 2, \quad \text{or for any } \alpha \geq 0 \text{ if } d = 2, \tag{2.7}$$

$$|\varepsilon(\mathbf{x},k)| \leq C_\varepsilon|k|^{\theta} \quad \text{for} \quad 0 \leq \theta < \frac{d}{d-2} \text{ if } d \neq 2, \quad \text{or for any } \theta \geq 0 \text{ if } d = 2 \tag{2.8}$$

for a.a. $\mathbf{x} \in \Omega$ and for some nonnegative constants C_f and C_ε. In this work we will consider the case when the feedbacks $\mathbf{f}(\mathbf{x},\mathbf{u})$ and $\varepsilon(\mathbf{x},k)$ are strong nonlinearities, *i.e.* when no upper restrictions on the growth of $\mathbf{f}(\mathbf{x},\mathbf{u})$ with respect to $\mathbf{u}$, and of $\varepsilon(\mathbf{x},k)$ with respect to k, such as (2.7)-(2.8), are required.

As in [20], we shall assume on the production term $P(\mathbf{x},\mathbf{u},k)$ the possibilities arising in the applications aforementioned. We consider the cases of

$$P(\mathbf{x},\mathbf{u},k) = \pi(\mathbf{x},\mathbf{u}) \quad \text{or} \quad P(\mathbf{x},\mathbf{u},k) = \varpi(\mathbf{x},\mathbf{u})k, \tag{2.9}$$

where, accordingly to (2.3),

$$\pi,\ \varpi : \Omega \times \mathbb{R}^d \to \mathbb{R} \quad \text{are Carathéodory functions.} \tag{2.10}$$

If $P(\mathbf{x},\mathbf{u},k) = \pi(\mathbf{x},\mathbf{u})$, we assume the existence of a nonnegative constant C_π such that

$$|\pi(\mathbf{x},\mathbf{u})| \leq C_\pi|\mathbf{u}|^{\beta} \quad \text{for} \quad 0 \leq \beta \leq \frac{d+2}{d-2} \ \text{ if } d \neq 2, \quad \text{or for} \quad \text{any } \beta \geq 0 \text{ if } d = 2, \tag{2.11}$$

for a.a. $\mathbf{x} \in \Omega$, whereas for $P(\mathbf{x},\mathbf{u},k) = \varpi(\mathbf{x},\mathbf{u})k$, we assume the existence of a positive constant C_ϖ such that

$$|\varpi(\mathbf{x},\mathbf{u})| \leq C_\varpi|\mathbf{u}|^{\beta} \quad \text{for} \quad 0 \leq \beta < \frac{4}{d-2} \text{ if } d \neq 2, \quad \text{or for any } \beta \geq 0 \text{ if } d = 2, \tag{2.12}$$

for a.a. $\mathbf{x} \in \Omega$.

Remark 2.3. *Conditions (2.11) and (2.12) generalize the models [16, 21] to encompass other situations of real-world turbulent flows (see [20] and the references cited therein). However and only for mathematical interest we may generalize even more these conditions as follows,*

$$P(\mathbf{x},\mathbf{u},k) = \begin{cases} \varpi(\mathbf{x},\mathbf{u})k^{2\sigma+1} & \text{if} \quad \sigma > -\frac{1}{2} \\ \pi(\mathbf{x},\mathbf{u}) & \text{if} \quad \sigma = -\frac{1}{2} \end{cases} \tag{2.13}$$

for $\pi(\mathbf{x},\mathbf{u})$ satisfying to (2.11) and for $\varpi(\mathbf{x},\mathbf{u})$ satisfying to

$$\begin{aligned} |\varpi(\mathbf{x},\mathbf{u})| \leq C_\varpi|\mathbf{u}|^{\beta} \quad \text{for} \quad \beta + 2\sigma < \frac{4}{d-2} \quad \text{and} \quad \beta \geq 0 \quad \text{if } d \neq 2, \\ \text{or} \quad \text{for any} \quad \beta \geq 0 \quad \text{if } d = 2, \end{aligned} \tag{2.14}$$

for a.a. $\mathbf{x} \in \Omega$. With not so difficulty changes in the proofs, the result of this work still holds as long as (2.11) and (2.14) hold and $-\frac{1}{2} \leq \sigma \leq 0$ in (2.13).

As in [20], and in order to be physically realistic with the turbulent models [16, 21], we need also to assume that

$$\pi(\mathbf{x},\mathbf{u}) \geq 0 \quad \text{and} \quad \varpi(\mathbf{x},\mathbf{u}) \geq 0 \quad \text{for all } \mathbf{u} \in \mathbb{R}^d \text{ and for a.a } \mathbf{x} \in \Omega. \tag{2.15}$$

On the feedback functions that account for the extra forcing term and for the turbulent dissipation, we assume the following sign conditions which in fact follow from the physics of the problem,

$$\mathbf{f}(\mathbf{x},\mathbf{u}) \cdot \mathbf{u} \geq 0 \quad \text{for all } \mathbf{u} \in \mathbb{R}^d \text{ and for a.a. } \mathbf{x} \in \Omega, \tag{2.16}$$

$$\varepsilon(\mathbf{x},k)\,k \geq 0 \quad \text{for all } k \in \mathbb{R} \text{ and for a.a. } \mathbf{x} \in \Omega. \tag{2.17}$$

Note that conditions (2.16)-(2.17) are satisfied by any feedback forces field and by any turbulent dissipation function used in the applications mentioned above (see [20] and the references cited therein). On the other hand, $k \geq 0$ in the physical situation, and the best known expressions for turbulent dissipation, turbulent viscosity and turbulent diffusion functions are given by the Prandtl model, when giving, for instance, by the following expressions

$$\varepsilon(\mathbf{x},k) = \frac{k\sqrt{k}}{l(\mathbf{x})}, \qquad \nu_T(\mathbf{x},k) = C_1 l(\mathbf{x})\sqrt{k}, \qquad \nu_D(\mathbf{x},k) = \mu_e + C_2 l(\mathbf{x})\sqrt{k}, \qquad l \neq 0, \quad k \geq 0, \tag{2.18}$$

where μ_e is an effective (dynamic) viscosity, C_1, C_2 are dimensionless constants and $l : \Omega \to \mathbb{R}$ is the mixing length function which is usually assumed to satisfy $l(\mathbf{x}) \geq l_0$ for a.a. $\mathbf{x} \in \Omega$ and for some positive constant l_0 (see *e.g.* [7, 15, 18]). Motivated by this, we assume that our general turbulent dissipation function can be written in such a way that

$$\varepsilon(\mathbf{x}, k) = k e(\mathbf{x}, k) \qquad \text{where} \quad e : \Omega \times \mathbb{R} \to \mathbb{R} \text{ is a Carathéodory function.} \tag{2.19}$$

Gathering the information of (2.17) and (2.19) it follows immediately that

$$e(\mathbf{x}, k) \geq 0 \quad \text{for all } k \in \mathbb{R} \text{ and for a.a. } \mathbf{x} \in \Omega\,. \tag{2.20}$$

Note that, in the particular case of the Prandtl formula $(2.18)_1$, the function $e(\mathbf{x}, k) = \frac{\sqrt{k}}{l(\mathbf{x})}$ satisfies to (2.20) whenever $k \geq 0$ and $l(\mathbf{x}) \geq l_0$ for a.a. $\mathbf{x} \in \Omega$ and for some positive constant l_0. Observe that in this work we are considering a much more general situation in which the function $e(\mathbf{x}, k)$ is assumed to be a strong nonlinearity.

There is another set of assumptions, already touched on at (2.4), that are mathematically needed,

$$0 \leq \nu_T(\mathbf{x}, k) \leq C_T \quad \text{for all } k \in \mathbb{R} \text{ and for a.a. } \mathbf{x} \in \Omega, \quad C_T \in \mathbb{R}^+, \tag{2.21}$$

$$0 < c_D \leq \nu_D(\mathbf{x}, k) \leq C_D \quad \text{for all } k \in \mathbb{R} \text{ and for a.a. } \mathbf{x} \in \Omega, \quad c_D, C_D \in \mathbb{R}^+. \tag{2.22}$$

To avoid the trivial solution $k = 0$, we shall assume in the sequel, and in addition to (2.21), that

$$\nu_T(\mathbf{x}, k) \neq 0 \quad \text{when } k = 0\,. \tag{2.23}$$

Because we are considering a very general model that encompasses other situations of turbulence modelling, the novelty of this work is threefold. On one hand, our result will improve the existence theory for the usual turbulent $k-$epsilon model, *i.e.* when we consider $\mathbf{f}(\mathbf{x}, \mathbf{u}) = \mathbf{0}$ and $P(\mathbf{x}, \mathbf{u}, k) = 0$ in the model problem (1.1)-(1.4), but with no upper restriction on the growth of $\varepsilon(\mathbf{x}, k)$ with respect to k. In this case, our work improves the results known in the literature for the steady case of the turbulent $k-$epsilon model. On the other hand, our work will also apply when $\mathbf{f}(\mathbf{x}, \mathbf{u}) \neq \mathbf{0}$ and $P(\mathbf{x}, \mathbf{u}, k) = 0$, which is characteristic, for instance, of modelling turbulent flows in a rotating frame. A third and main application of our work, is for the case of both non-zero $\mathbf{f}(\mathbf{x}, \mathbf{u})$ and $P(\mathbf{x}, \mathbf{u}, k)$, characteristic of turbulent flows through porous media.

3. The main result

In this section, we state the existence result on which we are concerned in this work and we start here with the approach to prove it. For the sake of simplifying the writing, from now on, we shall no longer write the dependence of the Carathéodory functions (2.1)-(2.3) and (2.19) on the space variable $\mathbf{x}$. As we have mentioned before, the novelty of this work relies on the consideration of the feedback terms $\mathbf{f}(\mathbf{x}, \mathbf{u})$ and $\varepsilon(\mathbf{x}, k)$ as strong nonlinearities. For these terms, we only assume that

$$\exists\, \tau > 0 : |\angle(\mathbf{f}(\mathbf{u}), \mathbf{u})| \notin \left(\frac{\pi}{2} - \tau, \frac{\pi}{2} + \tau\right) \quad \forall\, \mathbf{u} : |\mathbf{u}| \geq L, \quad \forall\, L > 0, \tag{3.1}$$

$$H_L \in \mathbf{L}^1(\Omega) \quad \forall\, L > 0, \quad \text{where} \quad H_L := \sup_{|\mathbf{u}| \leq L} |\mathbf{f}(\mathbf{u})|, \tag{3.2}$$

$$G_M \in \mathrm{L}^1(\Omega) \quad \forall\, M > 0, \quad \text{where} \quad G_M := \sup_{|k| \leq M} |\varepsilon(k)|. \tag{3.3}$$

Theorem 3.1. *Let Ω be a bounded domain of $\mathbb{R}^d$, $2 \leq d \leq 4$, with a Lipschitz-continuous boundary $\partial\Omega$. Assume all the conditions (2.1)-(2.3), (2.9)-(2.10), (2.15), (2.16)-(2.17), (2.19)-(2.20), (2.21)-(2.23) and*

$$\mathbf{g} \in \mathbf{L}^2(\Omega). \tag{3.4}$$

In addition, assume that (3.1)-(3.3) hold and one of the following conditions is satisfied:

(1) $P(\mathbf{u}, k) = \pi(\mathbf{u})$ a.e. in Ω and (2.11) holds;
(2) $P(\mathbf{u}, k) = \varpi(\mathbf{u})k$ a.e. in Ω, (2.12) holds, and

$$c_D > C\left(\frac{\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}}{\nu}\right)^{\beta} \tag{3.5}$$

for the positive constant C defined at (4.18), and where ν is the positive constant that accounts for the kinematic viscosity.

Then there exists, at least, a weak solution to the problem (1.1)-(1.4). Moreover $\mathbf{f}(\mathbf{u}) \in \mathbf{L}^1(\Omega)$ *and* $\mathbf{f}(\mathbf{u})\cdot\mathbf{u}$, $\varepsilon(k)$, $\varepsilon(k)k \in \mathrm{L}^1(\Omega)$.

Remark 3.1. *Note that conditions (3.1)-(3.3), together with conditions (2.16)-(2.17), do not imply any upper restrictions on the growth of* $\mathbf{f}(\mathbf{u})$ *with respect to* $\mathbf{u}$ *nor of* $\varepsilon(k)$ *with respect to k. This is why, sometimes, this type of feedback terms are called as strong nonlinearities (see [2, 3, 4, 6]). We will handle these feedback terms by using a truncation argument whose application needs conditions (3.2)-(3.3) to control the behaviour of* $\mathbf{f}(\mathbf{u})$ *and of* $\varepsilon(k)$ *for large values of* $|\mathbf{u}|$ *and of k, respectively. On the other hand, condition (3.1), on the angle determined by the vectors* $\mathbf{f}(\mathbf{u})$ *and* $\mathbf{u}$*, denoted there by* $\angle(\mathbf{f}(\mathbf{u}),\mathbf{u})$*, says that* $\mathbf{f}(\mathbf{u})$ *and* $\mathbf{u}$ *cannot be nearly orthogonal for large values of* $|\mathbf{u}|$*. This condition is particularly useful to avoid the occurrence of feedbacks* $\mathbf{f}(\mathbf{u})$ *of the following Coriolis-Pohozhaev type*

$$\mathbf{f}(u,v) = e^{|(u,v)|^2}(-v,u), \quad \mathbf{u}=(u,v) \quad (d=2)$$

for which (2.16) is satisfied but $\mathbf{f}(\mathbf{u}) \notin \mathbf{L}^1(\Omega)$.

The proof of Theorem 3.1 is still started in the current section and it will be carried out through the rest of the paper, in Sections 4 and 5. But first we observe that several technical difficulties arise. Since we do not have any upper restriction on the growth of $\mathbf{f}(\mathbf{u})$ with respect to $\mathbf{u}$, we will use a truncation argument in the spirit of [2, 3] (see also [4, 6]). By the same reason, we will use a truncation argument in the term $\varepsilon(k)$ in the spirit of [4] (see also [6]). On the other hand, because $|\mathbf{D}(\mathbf{u})|^2$ is only in $\mathbf{L}^1(\Omega)$, we will use a regularization of this term in the spirit of [20] (see also [17]).

Proof. (of Theorem 3.1) We start by considering, for each $n \in \mathbb{N}$, the following truncated and regularized problem

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in} \quad \Omega, \tag{3.6}$$

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{g} - \mathbf{f}_n(\mathbf{u}) - \nabla p + \mathbf{div}\left((\nu+\nu_T(k))\,\mathbf{D}(\mathbf{u})\right) \quad \text{in } \Omega, \tag{3.7}$$

$$\mathbf{u}\cdot\nabla k = \operatorname{div}\left(\nu_D(k)\nabla k\right) + \nu_T(k)\,\mathcal{R}_n\Big(|\mathbf{D}(\mathbf{u})|^2\Big) - \varepsilon_n(k) + P(\mathbf{u},k) \quad \text{in } \Omega, \tag{3.8}$$

$$\mathbf{u} = 0 \quad \text{and} \quad k = 0 \quad \text{on } \partial\Omega, \tag{3.9}$$

where $\mathbf{f}_n(\mathbf{u})$ is the n-radial truncation of $\mathbf{f}(\mathbf{u})$,

$$\mathbf{f}_n(\mathbf{u}) := \begin{cases} \mathbf{f}(\mathbf{u}) & \text{if } |\mathbf{u}| \le n, \\ \mathbf{f}(\mathbf{u}^n) & \text{if } |\mathbf{u}| = \rho > n, \end{cases} \tag{3.10}$$

with $\mathbf{u}^n$ being such that $|\mathbf{u}^n| = n$, $\varepsilon_n(k)$ is the n-truncation of $\varepsilon(k)$

$$\varepsilon_n(k) := \begin{cases} \varepsilon(k) & \text{if } |k| \le n, \\ \varepsilon(k^n) & \text{if } |k| = \varrho > n, \end{cases} \tag{3.11}$$

where k^n satisfies to $|k^n| = n$. We recall that, due to the assumptions (2.1)-(2.2) and to the definitions (3.10)-(3.11), the following well-known facts hold (see *e.g.* [24, Lemma 3.4.3] and [6, Proposition 2]),

$$\mathbf{f}_n \text{ is a continuous function on } \mathbf{u} \text{ and } \quad \exists\, C_{\mathbf{f}_n} > 0 : |\mathbf{f}_n(\mathbf{u})| \le C_{\mathbf{f}_n} \quad \forall\, \mathbf{u} \in \mathbb{R}^d, \tag{3.12}$$

$$\varepsilon_n \text{ is a continuous function on } k \text{ and } \quad \exists\, C_{\varepsilon_n} > 0 : |\varepsilon_n(k)| \le C_{\varepsilon_n} \quad \forall\, k \in \mathbb{R}. \tag{3.13}$$

In (3.8), $\mathcal{R}_n(h)$ denotes a truncation of the term h such that

$$\mathcal{R}_n(h) := \frac{h}{1+\frac{1}{n}h}, \quad \text{where} \quad h = |\mathbf{D}(\mathbf{u})|^2. \tag{3.14}$$

Observe that the regularization $\mathcal{R}_n(h)$ satisfies to

$$\mathcal{R}_n(h) \le \min\{h,\, n\} \quad \forall\, h \ge 0. \tag{3.15}$$

In the conditions of Theorem 3.1, we say that a pair $(\mathbf{u},k)$ is a weak solution to the problem (3.6)-(3.9) if, for each $n \in \mathbb{N}$,

(1') $\mathbf{u} \in \mathbf{V}$ and for every $\mathbf{v} \in \mathbf{V} \cap \mathbf{L}^d(\Omega)$ there holds

$$\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\,d\mathbf{x} + \int_\Omega (\nu+\nu_T(k))\,\mathbf{D}(\mathbf{u}) : \nabla\mathbf{v}\,d\mathbf{x} + \int_\Omega \mathbf{f}_n(\mathbf{u})\cdot\mathbf{v}\,d\mathbf{x} = \int_\Omega \mathbf{g}\cdot\mathbf{v}\,d\mathbf{x}; \tag{3.16}$$

(2') $k \in \mathrm{H}^1_0(\Omega)$ and for every $\varphi \in \mathrm{H}^1_0(\Omega) \cap \mathrm{L}^d(\Omega)$ there holds

$$\int_\Omega (\mathbf{u}\cdot\boldsymbol{\nabla} k)\varphi\, d\mathbf{x} + \int_\Omega \nu_D(k)\boldsymbol{\nabla} k\cdot\boldsymbol{\nabla}\varphi\, d\mathbf{x} + \int_\Omega \varepsilon_n(k)\,\varphi\, d\mathbf{x} = \int_\Omega \nu_T(k)\mathcal{R}_n\Big(|\mathbf{D}(\mathbf{u})|^2\Big)\varphi\, d\mathbf{x} + \int_\Omega P(\mathbf{u},k)\,\varphi\, d\mathbf{x}; \tag{3.17}$$

(3') $k \geq 0$ and $\varepsilon_n(k) \geq 0$ *a.e.* in Ω.

Remark 3.2. *Note that in the case of $2 \leq d \leq 4$, we are considering in this work, the Sobolev imbedding $\mathbf{H}^1_0(\Omega) \hookrightarrow \mathbf{L}^d(\Omega)$ holds and therefore it is only needed to require the test functions of (3.16) are in the function space $\mathbf{V}$. Analogously the test functions of (3.17) are only required to be in the function space $\mathrm{H}^1_0(\Omega)$.*

Proposition 3.1. *Let the conditions of Theorem 3.1 be fulfilled. Then (for each $n \in \mathbb{N}$) there exists, at least, a weak solution to the problem (3.6)-(3.9) satisfying to (1')-(3') above.*

In the sequel and in order to optimize some estimates that are needed, we will make use of the best constants for the Poincaré and Sobolev inequalities. We start by recalling that the principal (positive) eigenvalue, say $\lambda_P(d)$, for the Laplacian problem

$$\begin{cases} \Delta\phi = -\lambda\phi & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega \end{cases} \tag{3.18}$$

is attained in such a way that $0 < \lambda_P(d) < \infty$. Moreover, $\lambda_P(d)$ is the best possible constant in the Poincaré's inequality,

$$\|\phi\|^2_{\mathrm{L}^2(\Omega)} \leq \lambda_P(d)\|\boldsymbol{\nabla}\phi\|^2_{\mathrm{L}^2(\Omega)} \quad \forall\, \phi \in \mathrm{H}^1_0(\Omega). \tag{3.19}$$

In order to simplify the exposition, in the sequel we rename the positive constant $\lambda_P(d)$ considered in (3.19) as $\lambda_P(d)^2$. The extension of (3.19) to a general L^r norm, making use or not of the Sobolev inequality, was studied by many authors. In particular, for $r = 2$, the sharpest constant of the Sobolev inequality, say $\lambda(2,d)$, is attained in such a way that $0 < \lambda(2,d) < \infty$ and

$$\|\phi\|_{\mathrm{L}^{2^*}(\Omega)} \leq \lambda(2,d)\|\boldsymbol{\nabla}\phi\|_{\mathrm{L}^2(\Omega)} \quad \forall\, \phi \in \mathrm{H}^1_0(\Omega), \quad \forall d \geq 2\,, \tag{3.20}$$

where 2^* denotes the Sobolev conjugate of 2, i.e. $2^* = \frac{2d}{d-2}$ if $d \neq 2$ and 2^* is any real in the interval $[1,\infty)$ otherwise. See [20] and the references cited therein for a better understanding. In the sequel we shall use capital letters, $\Lambda_P(d)$ and $\Lambda(2,d)$ for the best constants of the vectorial versions of (3.19) and (3.20).

The proof of Proposition 3.1 is organized in several steps in the next section.

4. Proof of Proposition 3.1

4.1. **Existence of approximate solutions.** Let $\{(\mathbf{v}_i,\upsilon_i)\}_{i=1}^\infty$ be a set of non-trivial solutions $(\mathbf{v}_i,\upsilon_i)$, associated to the eigenvalues $\Lambda_i > 0$ and $\lambda_i > 0$, $i = 1,2,...$, of the following spectral problems,

$$\begin{cases} \displaystyle\sum_{|\alpha|=1}\int_\Omega D^\alpha\mathbf{v}_i\cdot D^\alpha\mathbf{w}\, d\mathbf{x} = \Lambda_i\int_\Omega \mathbf{v}_i\cdot\mathbf{w}\, d\mathbf{x} \qquad \forall\, \mathbf{w} \in \mathbf{V}, \\ \mathbf{v}_i \in \mathbf{V}, \end{cases}$$

$$\begin{cases} \displaystyle\sum_{|\alpha|=1}\int_\Omega D^\alpha\upsilon_i D^\alpha\omega\, d\mathbf{x} = \lambda_i\int_\Omega \upsilon_i\omega\, d\mathbf{x} \quad \forall\, \omega \in V, \\ \upsilon_i \in V. \end{cases}$$

The family $\{\mathbf{v}_i\}_{i=1}^\infty$ is orthogonal in $\mathbf{V}$ and can be chosen as being orthonormal in $\mathbf{H}$ (see *e.g.* [23]), whereas the family $\{\upsilon_i\}_{i=1}^\infty$ is orthogonal in V and can be chosen as being orthonormal in $\mathrm{L}^2(\Omega)$ (see *e.g.* [8]). Given $j \in \mathbb{N}$, let us consider the correspondingly j-dimensional spaces $\mathbf{V}^j$ and V^j spanned by $\mathbf{v}_1,\mathbf{v}_2,...,\mathbf{v}_j$ and by $\upsilon_1,\upsilon_2,...,\upsilon_j$, respectively. For each $j \in \mathbb{N}$, we search for an approximate solution $(\mathbf{u}_j,k_j)$ of (3.16)-(3.17) in the form

$$\mathbf{u}_j = \sum_{i=1}^j c_{ij}\mathbf{v}_i, \qquad c_{ij} \in \mathbb{R}, \quad \mathbf{v}_i \in \mathbf{V}^j, \tag{4.1}$$

$$k_j = \sum_{i=1}^j d_{ij}\upsilon_i, \qquad d_{ij} \in \mathbb{R}, \quad \upsilon_i \in V^j\,. \tag{4.2}$$

These functions are found by solving the following system of $2j$ nonlinear algebraic equations, with respect to the $2j$ unknowns $c_{1j},\ c_{2j},\ \dots,\ c_{jj}$ and $d_{1j},\ d_{2j},\ \dots,\ d_{jj}$ obtained from (3.16) and from (3.17), respectively:

$$\begin{aligned}&\int_\Omega((\mathbf{u}_j\cdot\boldsymbol{\nabla})\mathbf{u}_j)\cdot\mathbf{v}_i\,d\mathbf{x}+\int_\Omega(\nu+\nu_T(k_j))\mathbf{D}(\mathbf{u}_j):\boldsymbol{\nabla}\mathbf{v}_i\,d\mathbf{x}+\int_\Omega\mathbf{f}_n(\mathbf{u}_j)\cdot\mathbf{v}_i\,d\mathbf{x}\\&=\int_\Omega\mathbf{g}\cdot\mathbf{v}_i\,d\mathbf{x}\quad\text{for } i=1,\ \dots,\ j;\end{aligned}\tag{4.3}$$

$$\begin{aligned}&\int_\Omega(\mathbf{u}_j\cdot\boldsymbol{\nabla}k_j)\upsilon_i\,d\mathbf{x}+\int_\Omega\nu_D(k_j)\boldsymbol{\nabla}k_j\cdot\boldsymbol{\nabla}\upsilon_i\,d\mathbf{x}+\int_\Omega\varepsilon_n(k_j)\upsilon_i\,d\mathbf{x}\\&=\int_\Omega\nu_T(k_j)\mathcal{R}_n\Big(|\mathbf{D}(\mathbf{u}_j)|^2\Big)\upsilon_i\,d\mathbf{x}+\int_\Omega P(\mathbf{u}_j,k_j)\upsilon_i\,d\mathbf{x},\quad i=1,\dots,j.\end{aligned}\tag{4.4}$$

Due to the assumptions (2.16)-(2.17), (2.21)-(2.22) and (3.4), we can use a variant of Brower's theorem (see *e.g.* [23, Lemma II.1.4]) to prove the existence of, at least, a solution to the system formed by (4.1)-(4.2) and (4.3)-(4.4). To do it so, we consider a function $\mathcal{P}$, from $\mathbf{V}^j\times V^j$ into itself defined in such a way that

$$\begin{aligned}&\mathcal{P}(\mathbf{v},\upsilon)\cdot(\mathbf{v},\upsilon):=\\&\int_\Omega((\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v})\cdot\mathbf{v}\,d\mathbf{x}+\int_\Omega(\nu+\nu_T(\upsilon))\mathbf{D}(\mathbf{v}):\boldsymbol{\nabla}\mathbf{v}\,d\mathbf{x}+\int_\Omega\mathbf{f}_n(\mathbf{v})\cdot\mathbf{v}\,d\mathbf{x}-\int_\Omega\mathbf{g}\cdot\mathbf{v}\,d\mathbf{x}+\\&\int_\Omega(\mathbf{v}\cdot\boldsymbol{\nabla}\upsilon)\upsilon\,d\mathbf{x}+\int_\Omega\nu_D(\upsilon)|\boldsymbol{\nabla}\upsilon|^2\,d\mathbf{x}+\int_\Omega\varepsilon_n(\upsilon)\upsilon\,d\mathbf{x}-\int_\Omega\nu_T(\upsilon)\mathcal{R}_n\Big(|\mathbf{D}(\mathbf{v})|^2\Big)\upsilon\,d\mathbf{x}-\int_\Omega P(\mathbf{v},\upsilon)\upsilon\,d\mathbf{x}\\&:=I_1+\cdots-I_4+\cdots-I_8-I_9\end{aligned}\tag{4.5}$$

for all $(\mathbf{v},\upsilon)\in\mathbf{V}^j\times V^j$ and where the scalar product is induced by $\mathbf{V}\times V$. Evidently, $\mathcal{P}$ so defined is continuous. Since $\mathbf{v}\in\mathbf{V}^j$ implies $\mathbf{div}\,\mathbf{v}=0$, we have $I_1=0$ and $I_5=0$. On the other hand, arguing as we did in [20, Section 4.1], it can be proved that

$$\begin{aligned}&I_2\geq\nu C_K^2\|\boldsymbol{\nabla}\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)}\quad\text{and}\quad I_6\geq c_D\|\boldsymbol{\nabla}\upsilon\|^2_{\mathrm{L}^2(\Omega)},\\&I_4=\int_\Omega\mathbf{g}\cdot\mathbf{v}\,d\mathbf{x}\leq\Lambda_P(d)\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}\|\boldsymbol{\nabla}\mathbf{v}\|_{\mathbf{L}^2(\Omega)},\\&I_8=\int_\Omega\nu_T(\upsilon)\mathcal{R}_n\Big(|\mathbf{D}(\mathbf{v})|^2\Big)\upsilon\,d\mathbf{x}\leq C_T\,n\|\upsilon\|_{\mathrm{L}^1(\Omega)}\leq C_T\,n\sqrt{\mathcal{L}^d(\Omega)}\lambda(2,d)\|\boldsymbol{\nabla}\upsilon\|_{\mathrm{L}^2(\Omega)},\end{aligned}$$

where C_K is the Korn inequality's constant and $\mathcal{L}^d(\Omega)$ denotes the $d-$Lebesgue measure of Ω.

Now, let us show that

$$I_3\geq0\quad\text{and}\quad I_7\geq0$$

by proving that, in view of (2.16)-(2.17), we have

$$\mathbf{f}_n(\mathbf{u})\cdot\mathbf{u}\geq0\quad\forall\,\mathbf{u}\in\mathbb{R}^d,\tag{4.6}$$

$$\varepsilon_n(k)k\geq0\quad\forall\,k\in\mathbb{R}\tag{4.7}$$

a.e. in Ω. To prove (4.6), we use generalized spherical coordinates $(\rho,\theta_1,\dots,\theta_d)$ to write $\mathbf{u}=\rho\mathbf{w}$, where

$$\begin{aligned}&w_1:=\cos(\theta_1)\\&w_2:=\sin(\theta_1)\cos(\theta_2)\\&w_3:=\sin(\theta_1)\sin(\theta_2)\cos(\theta_3)\\&\vdots\\&w_{d-1}:=\sin(\theta_1)\cdots\sin(\theta_{d-2})\cos(\theta_{d-1})\\&w_d:=\sin(\theta_1)\cdots\sin(\theta_{d-2})\sin(\theta_{d-1})\,.\end{aligned}$$

Using the definition of the truncation (see (3.10)), we have, due to the assumption (2.16), that $\mathbf{f}_n(\mathbf{u})\cdot\mathbf{u}=\mathbf{f}(\mathbf{u})\cdot\mathbf{u}\geq0$ if $|\mathbf{u}|\leq n$. If $|\mathbf{u}|=\rho>n$, then $\mathbf{f}_n(\mathbf{u})\cdot\mathbf{u}=\rho\,\mathbf{f}(\mathbf{u}^n)\cdot\mathbf{w}$, and the later is nonnegative if $\mathbf{f}(\mathbf{u}^n)\cdot\mathbf{w}\geq0$. In fact, if $\mathbf{f}(\mathbf{u}^n)\cdot\mathbf{w}<0$, then we also would have $\mathbf{f}(\mathbf{u}^n)\cdot\mathbf{u}^n=n\mathbf{f}(\mathbf{u}^n)\cdot\mathbf{w}<0$ which contradicts the assumption (2.16). As for (4.7), it is a consequence of (2.17) and (3.11). In fact, if $|k|\leq n$ then $\varepsilon_n(k)k=\varepsilon(k)k\geq0$, by (2.17) and (3.11). If $|k|>n$ then

$\varepsilon_n(k)k = \varepsilon(k^n)k^n \times \frac{k}{k^n}$ due to (3.11). In this case, we observe, also from (3.11), that k and k^n are both nonzero and have the same sign. As a consequence of this and of (2.17), we readily have $\varepsilon_n(k)k \geq 0$ if $|k| > n$.

4.1.1. ***If*** $P(\mathbf{u}, k) = \pi(\mathbf{u})$ ***a.e. in*** Ω*,* we argue as we did in [20, Section 4.1.1], in particular by using (2.11) together with Hölder's inequality and both scalar and vectorial versions of Sobolev's inequality (3.20), to show that

$$I_9 = \int_\Omega \pi(\mathbf{v})\upsilon\, d\mathbf{x} \leq C_\pi\, \lambda(2,d)\Lambda(2,d)^\beta \|\nabla \mathbf{v}\|^\beta_{\mathbf{L}^2(\Omega)} \|\nabla \upsilon\|_{\mathrm{L}^2(\Omega)}. \tag{4.8}$$

Gathering the information of the estimates of $I_1, \ldots, I_9$ as we did in [20, Section 4.1.1], it follows from (4.5) that $\mathcal{P}(\mathbf{v}, \upsilon) \cdot (\mathbf{v}, \upsilon) > 0$ for $\|\mathbf{v}\|_{\mathbf{V}} = \rho$ and $\|\upsilon\|_V = \varsigma$, and where ρ and ς are chosen in such a way that

$$\rho > \frac{\Lambda_P(d)}{\nu C_K^2}\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)} \quad \text{and} \quad \varsigma > \frac{1}{c_D}\left(C_T\, n \sqrt{\mathcal{L}^d(\Omega)}\lambda(2,d) + C_\pi\, \lambda(2,d)\Lambda(2,d)^\beta \rho^\beta\right).$$

4.1.2. ***If*** $P(\mathbf{u}, k) = \varpi(\mathbf{u})k$ ***a.e. in*** Ω*,* we argue as we did in [20, Section 4.1.2] to show that

$$I_9 = \int_\Omega \varpi(\mathbf{v})\, \upsilon^2\, d\mathbf{x} \leq C_\varpi\, \lambda(2,d)^2 \Lambda(2,d)^\beta \|\nabla \mathbf{v}\|^\beta_{\mathbf{L}^2(\Omega)} \|\nabla \upsilon\|^2_{\mathrm{L}^2(\Omega)}, \tag{4.9}$$

by the application of (2.12) together with Hölder's inequality and both scalar and vectorial versions of Sobolev's inequality (3.20). In this case, gathering the information of the estimates of $I_1, \ldots, I_9$, it can be proved from (4.5) that $\mathcal{P}(\mathbf{v}, \upsilon) \cdot (\mathbf{v}, \upsilon) > 0$ for $\|\mathbf{v}\|_{\mathbf{V}} = \rho$ and $\|\upsilon\|_V = \varsigma$, and ρ and ς chosen in such a way that

$$\rho > \frac{\Lambda_P(d)}{\nu C_K^2}\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)} \quad \text{and} \quad \varsigma > \frac{C_T\, n \sqrt{\mathcal{L}^d(\Omega)}\lambda(2,d)}{c_D - C_\varpi\, \lambda(2,d)^2 \Lambda(2,d)^\beta \rho^\beta} \quad \text{and} \quad \varsigma > 0.$$

The hypotheses of [23, Lemma II.1.4] are thus verified and therefore there exists a solution $(\mathbf{c}_j, \mathbf{d}_j)$, with $\mathbf{c}_j := (c_{1j}, c_{2j}, \ldots, c_{jj})$ and $\mathbf{d}_j := (d_{1j}, d_{2j}, \ldots, d_{jj})$ to the system (4.1)-(4.4).

4.2. **A priori estimates.** Multiplying (4.3) by c_{ij}, adding up the resulting equation between $i = 1$ and $i = j$, and observing that the convective integral term vanishes due to the fact that $\operatorname{div} \mathbf{u}_j = 0$, we obtain

$$\int_\Omega (\nu + \nu_T(k_j))\mathbf{D}(\mathbf{u}_j) : \nabla \mathbf{u}_j\, d\mathbf{x} + \int_\Omega \mathbf{f}_n(\mathbf{u}_j) \cdot \mathbf{u}_j\, d\mathbf{x} = \int_\Omega \mathbf{g} \cdot \mathbf{u}_j\, d\mathbf{x}.$$

Using the symmetry of $\mathbf{D}(\mathbf{u}_j)$ together with the sign property (4.6) and with the assumption (2.21), we get

$$\nu \int_\Omega |\mathbf{D}(\mathbf{u}_j)|^2\, d\mathbf{x} \leq \int_\Omega \mathbf{g} \cdot \mathbf{u}_j\, d\mathbf{x}.$$

Then, arguing as we did in [20, Section 4.2], it can be proved that

$$\|\nabla \mathbf{u}_j\|_{\mathbf{L}^2(\Omega)} \leq C, \quad C = C(\nu, d, \Omega, \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}), \tag{4.10}$$

where the positive constant C is independent of j. As a consequence of (4.10), we have, up to some subsequences,

$$\mathbf{u}_j \to \mathbf{u} \quad \text{weakly in } \mathbf{H}^1_0(\Omega), \quad \text{as } j \to \infty, \tag{4.11}$$

$$\mathbf{u}_j \to \mathbf{u} \quad \text{strongly in } \mathbf{L}^\gamma(\Omega), \quad \text{as } j \to \infty, \quad \text{for } \gamma \in [1, 2^*), \tag{4.12}$$

$$\mathbf{u}_j \to \mathbf{u} \quad \text{a.e. in } \Omega, \quad \text{as } j \to \infty. \tag{4.13}$$

In particular, a refinement of the estimate (4.10) allows us to write

$$\|\nabla \mathbf{u}_j\|_{\mathbf{L}^2(\Omega)} \leq \frac{\Lambda_P(d)}{\nu C_K^2}\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}, \tag{4.14}$$

and, observing the weak convergence (4.11), we obtain, by passing to the limit inf in (4.14), that

$$\|\nabla \mathbf{u}\|_{\mathbf{L}^2(\Omega)} \leq \frac{\Lambda_P(d)}{\nu C_K^2}\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}. \tag{4.15}$$

We will use this estimate later on, for the analysis of the equation for k, when the production term represented by the function P depend on both $\mathbf{u}$ and k.

Then we multiply (4.4) by d_{ij}, we add up the resulting equation between $i = 1$ and $i = j$ and we use the fact that $\mathbf{u}_j \in \mathbf{V}^j$ implies $\operatorname{div}\mathbf{u}_j = 0$ and whence the first integral term vanishes. Next we use the sign property (4.7) and we obtain,

$$\int_\Omega \nu_D(k_j)|\nabla k_j|^2\,d\mathbf{x} \le \int_\Omega \nu_T(k_j)\,\mathcal{R}_n\Big(|\mathbf{D}(\mathbf{u}_j)|^2\Big)\,k_j\,d\mathbf{x} + \int_\Omega P(\mathbf{u}_j,k_j)k_j\,d\mathbf{x}.$$

4.2.1. ***If*** $P(\mathbf{u},k) = \pi(\mathbf{u})$ ***a.e. in*** Ω, we proceed as in [20, Section 4.2.1] to obtain

$$\|\nabla k_j\|^2_{\mathrm{L}^2(\Omega)} \le C, \quad C = C(C_\pi, c_D, C_T, n, \Omega, d, \nu, \beta, \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}), \tag{4.16}$$

for a positive constant C not depending on j.

4.2.2. ***If*** $P(\mathbf{u},k) = \varpi(\mathbf{u})k$ ***a.e. in*** Ω, we argue as we did in [20, Section 4.2.2] to show that

$$\|\nabla k_j\|_{\mathrm{L}^2(\Omega)} \le C_T\, n\,\sqrt{\mathcal{L}^d(\Omega)}\lambda(2,d)\left(c_D - C_\varpi\,\lambda(2,d)^2\Lambda(2,d)^\beta\left(\frac{\Lambda_P(d)}{\nu C_K^2}\right)^\beta \|\mathbf{g}\|^\beta_{\mathbf{L}^2(\Omega)}\right)^{-1}. \tag{4.17}$$

We can readily see that the right-hand side of (4.17) is a positive constant independent of j as long as (3.5) holds with

$$C := C_\varpi\,\lambda(2,d)^2\Lambda(2,d)^\beta C_K^{-2\beta}\Lambda_P(d)^\beta. \tag{4.18}$$

As a consequence of (4.16) or (4.17), it follows (up to some subsequences) that

$$k_j \to k \quad \text{weakly in } \mathrm{H}^1_0(\Omega), \quad \text{as } j\to\infty, \tag{4.19}$$

$$k_j \to k \quad \text{strongly in } \mathrm{L}^\gamma(\Omega), \quad \text{as } j\to\infty, \quad \text{for } \gamma\in[1,2^*), \tag{4.20}$$

$$k_j \to k \quad \text{a.e. in } \Omega, \quad \text{as } j\to\infty. \tag{4.21}$$

4.3. **Passing to the limit** $j \to \infty$**.** We start by passing to the limit $j\to\infty$ the integral equality (4.3). Arguing as we did in [20, Section 4.2], it can be proved from (4.10)-(4.13) that

$$\int_\Omega((\mathbf{u}_j\cdot\nabla)\mathbf{u}_j)\cdot\mathbf{v}_i\,d\mathbf{x} \to \int_\Omega((\mathbf{u}\cdot\nabla)\mathbf{u})\cdot\mathbf{v}_i\,d\mathbf{x}, \quad \text{as } j\to\infty, \quad \text{for all } i\ge 1, \tag{4.22}$$

$$\int_\Omega(\nu+\nu_T(k_j))\mathbf{D}(\mathbf{u}_j):\nabla\mathbf{v}_i\,d\mathbf{x} \to \int_\Omega(\nu+\nu_T(k))\mathbf{D}(\mathbf{u}):\nabla\mathbf{v}_i\,d\mathbf{x}, \quad \text{as } j\to\infty, \quad \text{for all } i\ge 1. \tag{4.23}$$

The only difference of (4.3) from the problem studied in [20] lies in the convergence of the third term, because in the current work we consider a strong nonlinearity $\mathbf{f}(\mathbf{u})$ that in this part of the proof is truncated as $\mathbf{f}_n(\mathbf{u})$. But, let us see that this convergence does not offer any difficult as well. In what follows, we just consider the case $d\ge 3$ (the case $d = 2$ is simpler). Using the first part of (3.12), we have by virtue of (4.13) that

$$\mathbf{f}_n(\mathbf{u}_j) \to \mathbf{f}_n(\mathbf{u}) \quad \text{a.e. in } \Omega, \quad \text{as } j\to\infty. \tag{4.24}$$

On the other hand, by the second part of (3.12), we have

$$\|\mathbf{f}_n(\mathbf{u}_j)\|_{\mathbf{L}^{\frac{2d}{d+2}}(\Omega)} \le C_{\mathbf{f}_n}(\mathcal{L}^d(\Omega))^{\frac{d+2}{2d}}. \tag{4.25}$$

Owing to (4.24) and (4.25), we obtain

$$\mathbf{f}(\mathbf{u}_j) \to \mathbf{f}(\mathbf{u}) \text{ weakly in } \mathbf{L}^{\frac{2d}{d+2}}(\Omega), \quad \text{as } j\to\infty, \tag{4.26}$$

By the Sobolev imbedding, $\mathbf{v}_i \in \mathbf{H}^1_0(\Omega) \hookrightarrow \mathbf{L}^{\frac{2d}{d-2}}(\Omega)$ and since $\left(\frac{2d}{d+2}\right)^{-1} + \left(\frac{2d}{d-2}\right)^{-1} = 1$, from (4.26), we have

$$\int_\Omega \mathbf{f}_n(\mathbf{u}_j)\cdot\mathbf{v}_i\,d\mathbf{x} \to \int_\Omega \mathbf{f}_n(\mathbf{u})\cdot\mathbf{v}_i\,d\mathbf{x}, \quad \text{as } j\to\infty, \quad \text{for all } i\ge 1. \tag{4.27}$$

Aside the convergence (4.27), we should mention here that the proof of (4.23) in [20] relies on the convergence

$$(\nu+\nu_T(k_j))\nabla\mathbf{v}_i \to (\nu+\nu_T(k))\nabla\mathbf{v}_i \quad \text{strongly in } \mathbf{L}^2(\Omega), \quad \text{as } j\to\infty, \tag{4.28}$$

which in turn can be proved by using (2.2), (2.21) and (4.21) together with Lebesgue's dominated convergence theorem.

The convergences (4.22), (4.23) and (4.27) imply that we can pass to the limit $j \to \infty$ in the approximate system (4.3) and thus we obtain

$$\begin{aligned}&\int_\Omega ((\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u})\cdot\mathbf{v}_i\,d\mathbf{x} + \int_\Omega (\nu+\nu_T(k))\mathbf{D}(\mathbf{u}) : \boldsymbol{\nabla}\mathbf{v}_i\,d\mathbf{x} + \int_\Omega \mathbf{f}_n(\mathbf{u})\cdot\mathbf{v}_i\,d\mathbf{x}\\ &= \int_\Omega \mathbf{g}\cdot\mathbf{v}_i\,d\mathbf{x}\end{aligned} \tag{4.29}$$

for all $i \geq 1$. Using the linearity of (4.29) in $\mathbf{v}_i$ and the density of the finite linear combinations of the system $\{\mathbf{v}_i\}_{i=1}^\infty$ in $\mathbf{V}$, we deduce that (4.29) holds true in the whole space $\mathbf{V}$, that is

$$\begin{aligned}&\int_\Omega ((\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u})\cdot\mathbf{v}\,d\mathbf{x} + \int_\Omega (\nu+\nu_T(k))\mathbf{D}(\mathbf{u}) : \boldsymbol{\nabla}\mathbf{v}\,d\mathbf{x} + \int_\Omega \mathbf{f}_n(\mathbf{u})\cdot\mathbf{v}\,d\mathbf{x}\\ &= \int_\Omega \mathbf{g}\cdot\mathbf{v}\,d\mathbf{x}\end{aligned} \tag{4.30}$$

for all $\mathbf{v} \in \mathbf{V}$. This allows us to take $\mathbf{v} = \mathbf{u}$ as a test function in (4.30), which yields

$$\int_\Omega (\nu+\nu_T(k))|\mathbf{D}(\mathbf{u})|^2\,d\mathbf{x} + \int_\Omega \mathbf{f}_n(\mathbf{u})\cdot\mathbf{u}\,d\mathbf{x} = \int_\Omega \mathbf{g}\cdot\mathbf{u}\,d\mathbf{x}. \tag{4.31}$$

On the other hand, taking $\mathbf{v}_i = \mathbf{u}_j$ in (4.3), we also have the equality

$$\int_\Omega (\nu+\nu_T(k_j))|\mathbf{D}(\mathbf{u}_j)|^2\,d\mathbf{x} + \int_\Omega \mathbf{f}_n(\mathbf{u}_j)\cdot\mathbf{u}_j\,d\mathbf{x} = \int_\Omega \mathbf{g}\cdot\mathbf{u}_j\,d\mathbf{x}. \tag{4.32}$$

In (4.31)-(4.32), we have used the facts that $\mathbf{u}$ and $\mathbf{u}_j$ are solenoidal and $\mathbf{D}(\mathbf{u})$ and $\mathbf{D}(\mathbf{u}_j)$ are symmetric. Then, using (4.31) and (4.32) together with (4.12), (4.20) and (4.27), we have

$$\lim_{j\to\infty}\int_\Omega (\nu+\nu_T(k_j))|\mathbf{D}(\mathbf{u}_j)|^2\,d\mathbf{x} = \int_\Omega \mathbf{g}\cdot\mathbf{u}\,d\mathbf{x} - \int_\Omega \mathbf{f}_n(\mathbf{u})\cdot\mathbf{u}\,d\mathbf{x} = \int_\Omega (\nu+\nu_T(k))|\mathbf{D}(\mathbf{u})|^2\,d\mathbf{x}. \tag{4.33}$$

On the other hand, arguing as we did for (4.23), we can prove that

$$(\nu+\nu_T(k_j))^{\frac{1}{2}}\mathbf{D}(\mathbf{u}_j) \to (\nu+\nu_T(k))^{\frac{1}{2}}\mathbf{D}(\mathbf{u}) \quad \text{weakly in } \mathbf{L}^2(\Omega), \quad \text{as } j\to\infty. \tag{4.34}$$

Combining (4.33) and (4.34), it yields

$$(\nu+\nu_T(k_j))^{\frac{1}{2}}\mathbf{D}(\mathbf{u}_j) \to (\nu+\nu_T(k))^{\frac{1}{2}}\mathbf{D}(\mathbf{u}) \quad \text{strongly in } \mathbf{L}^2(\Omega), \quad \text{as } j\to\infty. \tag{4.35}$$

Now, we observe that, in view of (2.21), we have

$$\begin{aligned}&\int_\Omega |\mathbf{D}(\mathbf{u}_j)-\mathbf{D}(\mathbf{u})|^2\,d\mathbf{x} \leq\\ &\frac{1}{\nu}\left[\int_\Omega (\nu+\nu_T(k_j))|\mathbf{D}(\mathbf{u}_j)|^2\,d\mathbf{x} - 2\int_\Omega (\nu+\nu_T(k_j))\mathbf{D}(\mathbf{u}_j):\mathbf{D}(\mathbf{u})\,d\mathbf{x} + \int_\Omega (\nu+\nu_T(k_j))|\mathbf{D}(\mathbf{u})|^2\,d\mathbf{x}\right].\end{aligned}$$

Then, using (4.35) in the first term, (4.34) in the second and reasoning, in the third term, as we did at (4.28), we can prove that

$$\mathbf{D}(\mathbf{u}_j) \to \mathbf{D}(\mathbf{u}) \quad \text{strongly in } \mathbf{L}^2(\Omega), \quad \text{as } j\to\infty. \tag{4.36}$$

Finally, by Riesz-Fisher's theorem, we have, up to a subsequence,

$$\mathbf{D}(\mathbf{u}_j) \to \mathbf{D}(\mathbf{u}) \quad \text{a.e. in } \Omega, \quad \text{as } j\to\infty. \tag{4.37}$$

We will now pass to the limit $j \to \infty$ the integral equality (4.4). Arguing in this part as we did in [20, Section 4.2], it can be proved from (4.10)-(4.13), (4.16) or (4.17), and (4.19)-(4.21) that

$$\int_\Omega (\mathbf{u}_j \cdot \boldsymbol{\nabla} k_j)\upsilon_i \, d\mathbf{x} \to \int_\Omega (\mathbf{u} \cdot \boldsymbol{\nabla} k)\upsilon_i \, d\mathbf{x}, \quad \text{as } j \to \infty, \quad \text{for all } i \geq 1, \tag{4.38}$$

$$\int_\Omega \nu_D(k_j)\boldsymbol{\nabla} k_j \cdot \boldsymbol{\nabla}\upsilon_i \, d\mathbf{x} \to \int_\Omega \nu_D(k)\boldsymbol{\nabla} k \cdot \boldsymbol{\nabla}\upsilon_i \, d\mathbf{x}, \quad \text{as } j \to \infty, \quad \text{for all } i \geq 1, \tag{4.39}$$

$$\int_\Omega \nu_T(k_j)\mathcal{R}_n\Big(|\mathbf{D}(\mathbf{u}_j)|^2\Big)\upsilon_i \, d\mathbf{x} \to \int_\Omega \nu_T(k)\mathcal{R}_n\Big(|\mathbf{D}(\mathbf{u})|^2\Big)\upsilon_i \, d\mathbf{x}, \quad \text{as } j \to \infty, \quad \text{for all } i \geq 1, \tag{4.40}$$

$$\int_\Omega \pi(\mathbf{u}_j)\upsilon_i \, d\mathbf{x} \to \int_\Omega \pi(\mathbf{u})\upsilon_i \, d\mathbf{x}, \quad \text{as } j \to \infty, \quad \text{for all } i \geq 1, \tag{4.41}$$

$$\int_\Omega \varpi(\mathbf{u}_j)k_j\upsilon_i \, d\mathbf{x} \to \int_\Omega \varpi(\mathbf{u})k\upsilon_i \, d\mathbf{x}, \quad \text{as } j \to \infty, \quad \text{for all } i \geq 1. \tag{4.42}$$

We only need to justify the convergence of the third term. Here again we just consider the case $d \geq 3$ (the case $d = 2$ is simpler). Due to the assumption of the first part of (3.13), we have, as a consequence of (4.21), that

$$\varepsilon_n(k_j) \to \varepsilon_n(k) \quad \text{a.e. in } \Omega, \quad \text{as } j \to \infty. \tag{4.43}$$

Using now the second part of (3.13), one gets

$$\|\varepsilon_n(k_j)\|_{\mathrm{L}^{\frac{2d}{d+2}}(\Omega)} \leq C_{\varepsilon_n}(\mathcal{L}^d(\Omega))^{\frac{d+2}{2d}}. \tag{4.44}$$

Owing to (4.43) and (4.44), we have

$$\varepsilon_n(k_j) \to \varepsilon_n(k) \quad \text{weakly in } \mathrm{L}^{\frac{2d}{d+2}}(\Omega), \quad \text{as } j \to \infty. \tag{4.45}$$

Then, since $\upsilon_i \in \mathrm{H}^1_0(\Omega) \hookrightarrow \mathrm{L}^{\frac{2d}{d-2}}(\Omega)$ and once that $\left(\frac{2d}{d+2}\right)^{-1} + \left(\frac{2d}{d-2}\right)^{-1} = 1$, we have, in view of (4.45), that

$$\int_\Omega \varepsilon_n(k_j)\upsilon_i \, d\mathbf{x} \to \int_\Omega \varepsilon_n(k)\upsilon_i \, d\mathbf{x}, \quad \text{as } j \to \infty, \quad \text{for all } i \geq 1. \tag{4.46}$$

The convergences (4.38), (4.39), (4.40), (4.46) and (4.41), or (4.42), assure that we can pass to the limit $j \to \infty$ in the approximate system (4.4) to obtain

$$\begin{aligned} &\int_\Omega (\mathbf{u} \cdot \boldsymbol{\nabla} k)\upsilon_i \, d\mathbf{x} + \int_\Omega \nu_D(k)\boldsymbol{\nabla} k \cdot \boldsymbol{\nabla}\upsilon_i \, d\mathbf{x} + \int_\Omega \varepsilon_n(k)\upsilon_i \, d\mathbf{x} \\ &= \int_\Omega \nu_T(k)\mathcal{R}_n\Big(|\mathbf{D}(\mathbf{u})|^2\Big)\upsilon_i \, d\mathbf{x} + \int_\Omega P(\mathbf{u}, k)\,\upsilon_i \, d\mathbf{x} \end{aligned} \tag{4.47}$$

for all $i \geq 1$.

We have thus proved that, for each $n \in \mathbb{N}$, there exists a weak solution $(\mathbf{u}_n, k_n) \in \mathbf{V} \times \mathrm{H}^1_0(\Omega)$ to the problem (3.6)-(3.9) and such that

$$\begin{aligned} &\int_\Omega (\mathbf{u}_n \cdot \boldsymbol{\nabla})\mathbf{u}_n \cdot \mathbf{v} \, d\mathbf{x} + \int_\Omega (\nu + \nu_T(k_n))\mathbf{D}(\mathbf{u}_n) : \boldsymbol{\nabla}\mathbf{v} \, d\mathbf{x} + \int_\Omega \mathbf{f}_n(\mathbf{u}_n) \cdot \mathbf{v} \, d\mathbf{x} \\ &= \int_\Omega \mathbf{g} \cdot \mathbf{v} \, d\mathbf{x} \end{aligned} \tag{4.48}$$

and

$$\begin{aligned} &\int_\Omega (\mathbf{u}_n \cdot \boldsymbol{\nabla} k_n)v \, d\mathbf{x} + \int_\Omega \nu_D(k_n)\boldsymbol{\nabla} k_n \cdot \boldsymbol{\nabla} v \, d\mathbf{x} + \int_\Omega \varepsilon_n(k_n)v \, d\mathbf{x} \\ &= \int_\Omega \nu_T(k_n)\mathcal{R}_n\Big(|\mathbf{D}(\mathbf{u}_n)|^2\Big)v \, d\mathbf{x} + \int_\Omega P(\mathbf{u}_n, k_n)\,v \, d\mathbf{x} \end{aligned} \tag{4.49}$$

hold for all $(\mathbf{v}, v) \in \mathbf{V}^j \times V^j$ and all $j \geq 1$. By linearity and density these relations hold for all $(\mathbf{v}, v) \in \mathbf{V} \times \mathrm{V}$, and by continuity they hold for all $(\mathbf{v}, v) \in \mathbf{V} \times \mathrm{H}^1_0(\Omega)$ due to the ranges of β set forth at (2.11)-(2.12).

4.4. **To show that $k_n \geq 0$ and $\varepsilon_n(k_n) \geq 0$.** In this part of the proof we need to introduce the typical form of the turbulent dissipation function used in the applications and that we have assumed in (2.19)-(2.20). Therefore, we consider the integral equation (4.49) written in the form

$$\begin{aligned}&\int_\Omega (\mathbf{u}_n\cdot\nabla k_n)v\,d\mathbf{x}+\int_\Omega \nu_D(k_n)\nabla k_n\cdot\nabla v\,d\mathbf{x}+\int_\Omega e_n(k_n)k_n v\,d\mathbf{x}\\ &=\int_\Omega \nu_T(k_n)\,\mathcal{R}_n\Big(|\mathbf{D}(\mathbf{u}_n)|^2\Big)\,v\,d\mathbf{x}+\int_\Omega P(\mathbf{u}_n,k_n)\,v\,d\mathbf{x},\end{aligned}\tag{4.50}$$

where

$$e_n(k):=\begin{cases} e(k) & \text{if } |k|\leq n,\\ e(k^n) & \text{if } |k|=\varrho>n,\end{cases}\tag{4.51}$$

being k^n such that $|k^n|=n$. From (2.19)-(2.20) and (4.51), it readily follows that

$$e_n(k)\geq 0\quad \text{for all } k\in\mathbb{R} \text{ and a.e. in } \Omega\,.\tag{4.52}$$

Observe that, in this particular case, (4.7) is immediately satisfied, because $\varepsilon_n(k)k=e_n(k)k^2\geq 0$, due to (3.11), (4.51) and (4.52)

Let us now consider a couple of functions $(\mathbf{u}_n,k_n)\in\mathbf{V}\times\mathrm{H}^1_0(\Omega)$ satisfying to (4.48) and (4.50). We start by decomposing k_n as $k_n=k_n^+-k_n^-$, where $k_n^+:=\max\{0,k_n\}$ and $k_n^-:=-\min\{0,k_n\}$. Since $k_n\in\mathrm{H}^1_0(\Omega)$ implies that $k_n^-\in\mathrm{H}^1_0(\Omega)$, we can take $v=-k_n^-$ in (4.50),

$$\begin{aligned}&-\int_\Omega (\mathbf{u}_n\cdot\nabla k_n)k_n^-\,d\mathbf{x}-\int_\Omega \nu_D(k_n)\nabla k_n\cdot\nabla k_n^-\,d\mathbf{x}-\int_\Omega e_n(k_n)k_nk_n^-\,d\mathbf{x}\\ &=-\int_\Omega \nu_T(k_n)\,\mathcal{R}_n\Big(|\mathbf{D}(\mathbf{u}_n)|^2\Big)\,k_n^-\,d\mathbf{x}-\int_\Omega P(\mathbf{u}_n,k_n)k_n^-\,d\mathbf{x}.\end{aligned}$$

Observing the properties of k_n^+ and k_n^- (see *e.g.* [7, p. 239]) and using the assumption (2.22) together with the fact that $\mathbf{u}_n\in\mathbf{V}$ implies $\operatorname{div}\mathbf{u}_n=0$, we obtain

$$c_D\int_\Omega|\nabla k_n^-|^2\,d\mathbf{x}+\int_\Omega e_n(k_n)(k_n^-)^2\,d\mathbf{x}\leq-\int_\Omega \nu_T(k_n)\,\mathcal{R}_n\Big(|\mathbf{D}(\mathbf{u}_n)|^2\Big)\,k_n^-\,d\mathbf{x}-\int_\Omega P(\mathbf{u}_n,k_n)k_n^-\,d\mathbf{x}.$$

Then (4.52) together with the assumption (2.21) and with the fact that $\mathcal{R}_n\Big(|\mathbf{D}(\mathbf{u}_n)|^2\Big)\geq 0$ (see (3.14)), yield

$$c_D\int_\Omega|\nabla k_n^-|^2\,d\mathbf{x}\leq-\int_\Omega P(\mathbf{u}_n,k_n)k_n^-\,d\mathbf{x}.\tag{4.53}$$

Then, using (2.15) and (2.22), and proceeding as we did in [20, Sections 4.4.1-2], we obtain

$$\|\nabla k_n^-\|^2_{\mathrm{L}^2(\Omega)}\leq 0\tag{4.54}$$

for either $P(\mathbf{u}_n,k_n)=\pi(\mathbf{u}_n)$ or $P(\mathbf{u}_n,k_n)=\varpi(\mathbf{u}_n)k_n$ a.e. in Ω. Note that, in the last case, conditions (2.12) and (3.5) are of the utmost importance. Then, by the Sobolev imbedding, $\|k_n^-\|^2_{\mathrm{L}^2(\Omega)}\leq 0$, which proves that $k_n^-=0$ a.e. in Ω and, consequently, $k_n\geq 0$ a.e. in Ω.

Then, as a direct consequence of applying the sign property (4.7) together with the conclusion that $k_n\geq 0$ a.e. in Ω, we have

$$\varepsilon_n(k_n)\geq 0\quad\text{a.e. in } \Omega.\tag{4.55}$$

The proof of Proposition 3.1 is now concluded.

5. End of the proof of Theorem 3.1

From Proposition 3.1, we know that, for each $n\in\mathbb{N}$, there exists a weak solution $(\mathbf{u}_n,k_n)\in\mathbf{V}\times\mathrm{H}^1_0(\Omega)$ to the problem (3.6)-(3.9) and such that (4.48)-(4.49) hold. The final part of the proof of Theorem 3.1 will be split into several parts for the sake of comprehension.

5.1. **A priori estimates.** We start by obtaining an estimate for $\mathbf{u}_n$. Since the sought solutions and the test functions are in the same function space, we can take $\mathbf{v} = \mathbf{u}_n$ in (4.48) and we obtain, after we use the symmetry of $\mathbf{D}(\mathbf{u}_n)$ and the fact that $\mathrm{div}\mathbf{u}_n = 0$,

$$\int_\Omega (\nu + \nu_T(k_n))|\mathbf{D}(\mathbf{u}_n)|^2\, d\mathbf{x} + \int_\Omega \mathbf{f}(\mathbf{u}_n)\cdot \mathbf{u}_n\, d\mathbf{x} = \int_\Omega \mathbf{g}\cdot\mathbf{u}_n\, d\mathbf{x}. \tag{5.1}$$

Proceeding as we did for (4.10) and (4.14), we obtain

$$\|\nabla \mathbf{u}_n\|_{\mathbf{L}^2(\Omega)} \leq \frac{C_K^2}{\nu}\Lambda_P(d)\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}. \tag{5.2}$$

Then, arguing as we did for (4.11)-(4.13) and in view of (5.2) and of the assumption (3.4), we have

$$\begin{aligned} &\mathbf{u}_n \to \mathbf{u} \quad \text{weakly in } \mathbf{H}_0^1(\Omega), \quad \text{as } n\to\infty,\\ &\mathbf{u}_n \to \mathbf{u} \quad \text{strongly in } \mathbf{L}^\gamma(\Omega), \quad \text{as } n\to\infty, \quad \text{for } \gamma\in[1,2^*),\\ &\mathbf{u}_n \to \mathbf{u} \quad \text{a.e. in } \Omega, \quad \text{as } n\to\infty \end{aligned} \tag{5.3}$$

To achieve an *a priori* estimate for k_n, we consider the following special test function in the spirit of [20] (see also [5, 17, 22]),

$$\varphi(k_n) := 1 - \frac{1}{(1+k_n)^\delta}, \quad \text{with} \quad \frac{(1+\delta)q}{2-q} \leq q*,$$

and where δ is a positive constant such that $\mathrm{W}^{1,q'}(\Omega) \hookrightarrow C^{0,\delta}(\Omega)$. These conditions on δ result that it must be $q' > d$. Observe that $\varphi(k_n)$ satisfies to

$$0 \leq \varphi(k_n) \leq 1, \quad \nabla\varphi(k_n) = \delta\frac{\nabla k_n}{(1+k_n)^{\delta+1}} \tag{5.4}$$

and therefore $\varphi(k_n) \in \mathrm{H}_0^1(\Omega)$. Thus we may take $v = \varphi(k_n)$ in (4.49) to get

$$\begin{aligned} &\int_\Omega (\mathbf{u}_n\cdot\nabla k_n)\varphi(k_n)\, d\mathbf{x} + \int_\Omega \nu_D(k_n)\nabla k_n\cdot\nabla\varphi(k_n)\, d\mathbf{x} + \int_\Omega \varepsilon_n(k_n)\varphi(k_n)\, d\mathbf{x}\\ &= \int_\Omega \nu_T(k_n)\,\mathcal{R}_n\Big(|\mathbf{D}(\mathbf{u}_n)|^2\Big)\,\varphi(k_n)\, d\mathbf{x} + \int_\Omega P(\mathbf{u}_n,k_n)\,\varphi(k_n)\, d\mathbf{x}, \end{aligned} \tag{5.5}$$

where the first term vanishes due to (3.6),

$$\int_\Omega (\mathbf{u}_n\cdot\nabla k_n)\varphi(k_n)\, d\mathbf{x} = \int_\Omega \mathbf{u}_n\cdot\nabla\Phi(k_n)\, d\mathbf{x} = 0, \quad \Phi(s) := \int_0^s \varphi(\tau)\, d\tau. \tag{5.6}$$

We observe that the third term of (5.5) is nonnegative due to (4.55) and to the fact that $\varphi(k_n)\geq 0$. As a consequence of this and of (5.6), and attending to (3.15) and (5.4), and there observing that $\varphi(k_n)\leq 1$, we obtain

$$\delta\int_\Omega \nu_D(k_n)\frac{|\nabla k_n|^2}{(1+k_n)^{1+\delta}}\, d\mathbf{x} \leq \int_\Omega \nu_T(k_n)\,|\mathbf{D}(\mathbf{u}_n)|^2\, d\mathbf{x} + \int_\Omega |P(\mathbf{u}_n,k_n)|\, d\mathbf{x}.$$

Then, proceeding as we did in [20, Sections 5.1.1-2], we obtain

$$\|\nabla k_n\|^q_{\mathrm{L}^q(\Omega)} \leq C, \quad C = C(\nu,\beta,c_D,C_T,C_\pi,d,q,\Omega,\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}) \quad \text{if } P(\mathbf{u},k) = \pi(\mathbf{u}) \text{ a.e. in } \Omega, \tag{5.7}$$

$$\|\nabla k_n\|^q_{\mathrm{L}^q(\Omega)} \leq C, \quad C = C(\nu,\beta,c_D,C_T,C_\varpi,d,q,\Omega,\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}) \quad \text{if } P(\mathbf{u},k) = \varpi(\mathbf{u})k \text{ a.e. in } \Omega. \tag{5.8}$$

Then, in view of (5.7) or (5.8), and of the assumption (3.4), we have (up to some subsequences)

$$k_n \to k \quad \text{weakly in } \mathrm{W}_0^{1,q}(\Omega), \quad \text{as } n\to\infty, \quad \text{for } q < d', \tag{5.9}$$

$$k_n \to k \quad \text{strongly in } \mathrm{L}^\gamma(\Omega), \quad \text{as } n\to\infty, \quad \text{for } \gamma\in[1,2^*), \tag{5.10}$$

$$k_n \to k \quad \text{a.e. in } \Omega, \quad \text{as } n\to\infty.$$

Now we can pass to the limit $n\to\infty$ almost integral terms of (4.48)-(4.49) by arguing analogously as we did in the previous section. The terms that require a special treatment are the ones involving $\mathbf{f}_n$, ε_n and $\mathcal{R}_n$, because we do not know wether if these terms remain bounded as $n\to\infty$. After the convergence of the term involving $\mathbf{f}_n$ has been established, the convergence of the term with $\mathcal{R}_n$ follows exactly as in [20, Section 5.2].

5.2. **Passing $\mathbf{f}_n(\mathbf{u}_n)$ to the limit $n \to \infty$.** Our aim is to show that

$$\int_\Omega \mathbf{f}_n(\mathbf{u}_n)\cdot\mathbf{v}\,d\mathbf{x} \to \int_\Omega \mathbf{f}(\mathbf{u})\cdot\mathbf{v}\,d\mathbf{x}, \quad \text{as } n\to\infty, \quad \forall\, \mathbf{v}\in\mathbf{V}\cap\mathbf{L}^\infty(\Omega). \tag{5.11}$$

To prove this, we shall make use of Vitali's convergence theorem. Let us first write

$$\mathbf{f}_n(\mathbf{u}_n)\cdot\mathbf{u}_n = |\mathbf{f}_n(\mathbf{u}_n)||\mathbf{u}_n|\cos(\theta_n), \quad \theta_n := \angle(\mathbf{f}_n(\mathbf{u}_n),\mathbf{u}_n),$$

and observe that, due to (5.1) and by using (5.2) together with the assumptions (2.21) and (3.4), it can be proved that

$$\int_\Omega \mathbf{f}_n(\mathbf{u}_n)\cdot\mathbf{u}_n\,d\mathbf{x} \le C, \tag{5.12}$$

for some positive constant C not depending on n. Then we observe that, according to (3.10) and (3.2), for any $L > 0$

$$|\mathbf{f}_n(\mathbf{u}_n)| = |\mathbf{f}(\mathbf{u}_n)| \le \sup_{|\mathbf{u}_n|<L}|\mathbf{f}(\mathbf{u}_n)| = H_L \quad \text{if } |\mathbf{u}_n| < L\,. \tag{5.13}$$

and, due to (5.12), we have

$$\mathrm{L}\int_\Omega |\mathbf{f}_n(\mathbf{u}_n)||\cos\theta_n|\,d\mathbf{x} \le \int_\Omega \mathbf{f}_n(\mathbf{u}_n)\cdot\mathbf{u}_n\,d\mathbf{x} \le C \quad \text{if } |\mathbf{u}_n| \ge L\,. \tag{5.14}$$

To use Vitali's theorem, it remains to show that $\{\mathbf{f}_n(\mathbf{u}_n)\}_{n\in\mathbb{N}}$ is uniformly integrable. For that, we consider an arbitrary $\epsilon > 0$ and an arbitrary measurable subset $E\subset\Omega$ such that $\mathcal{L}^d(E) < \delta$ for some $\delta > 0$. As a consequence of (5.13)-(5.14), we can write

$$\begin{aligned}\int_E |\mathbf{f}_n(\mathbf{u}_n)||\cos\theta_n|\,d\mathbf{x} &= \int_{E\cap\{|\mathbf{u}_n|<L\}} |\mathbf{f}_n(\mathbf{u}_n)||\cos\theta_n|\,d\mathbf{x} + \int_{E\cap\{|\mathbf{u}_n|\ge L\}} |\mathbf{f}_n(\mathbf{u}_n)||\cos\theta_n|\,d\mathbf{x}\\ &\le \int_{E\cap\{|\mathbf{u}_n|<L\}} |\mathbf{f}_n(\mathbf{u}_n)|\,d\mathbf{x} + \frac{C}{L} \le \int_E H_L\,d\mathbf{x} + \frac{C}{L}\end{aligned}$$

for any $L > 0$ and for some positive constant C not depending on n. Now, we take L such that $\frac{C}{L} < \frac{\epsilon}{2}$ and, by the assumption that $H_L\in\mathbf{L}^1(\Omega)$ (see (3.2)), we may choose δ in such a way that

$$\int_E H_L\,d\mathbf{x} \le \|H_L\|_{\mathbf{L}^1(\Omega)}\mathcal{L}_n(E) < \delta\|H_L\|_{\mathbf{L}^1(\Omega)} < \frac{\epsilon}{2}\,.$$

As a consequence, we have

$$\int_E |\mathbf{f}_n(\mathbf{u}_n)||\cos\theta_n|\,d\mathbf{x} < \epsilon, \tag{5.15}$$

and, by Vitali's theorem, $\mathbf{f}(\mathbf{u})\,\cos\theta\in\mathbf{L}^1(\Omega)$, where $\theta := \angle(\mathbf{f}(\mathbf{u}),\mathbf{u})$, and

$$\mathbf{f}_n(\mathbf{u}_n)\,\cos\theta_n \to \mathbf{f}(\mathbf{u})\,\cos\theta \quad \text{strongly in } \mathbf{L}^1(\Omega), \quad \text{as } n\to\infty.$$

On the other hand, by assumption (3.1),

$$|\cos\theta_n|,\ |\cos\theta| > \tau \quad \text{for any } n\in\mathbb{N},$$

for some $\tau > 0$, and, due to (5.15), we have

$$\int_E |\mathbf{f}_n(\mathbf{u}_n)|\,d\mathbf{x} < \frac{\epsilon}{\tau},$$

Then, again by Vitali's theorem, $\mathbf{f}(\mathbf{u})\in\mathbf{L}^1(\Omega)$ and

$$\mathbf{f}_n(\mathbf{u}_n) \to \mathbf{f}(\mathbf{u}) \quad \text{strongly in } \mathbf{L}^1(\Omega), \quad \text{as } n\to\infty. \tag{5.16}$$

Since the test function $\mathbf{v}\in\mathbf{V}\cap\mathbf{L}^\infty(\Omega)$, (5.11) follows from (5.16).

On the other hand, from (5.3) and (5.16), we have (up to some subsequences)

$$\mathbf{f}_n(\mathbf{u}_n)\cdot\mathbf{u}_n \to \mathbf{f}(\mathbf{u})\cdot\mathbf{u} \quad \text{a.e. in } \Omega, \quad \text{as } n\to\infty. \tag{5.17}$$

Then, due to (5.12) and (5.17), Fatou's lemma yields that $\mathbf{f}(\mathbf{u})\cdot\mathbf{u}\in\mathrm{L}^1(\Omega)$.

5.3. **Passing $\varepsilon_n(k_n)$ to the limit $n \to \infty$.** The proof of the convergence of the third term of (4.49) is analogous to previous one and even simpler, because here we are dealing with scalar functions. So, our aim here is to show that

$$\int_\Omega \varepsilon_n(k_n)\varphi\, d\mathbf{x} \to \int_\Omega \varepsilon(k)\varphi\, d\mathbf{x}, \quad \text{as } n \to \infty, \quad \forall\, \varphi \in W_0^{1,q'}(\Omega). \tag{5.18}$$

Similarly to the previous case, we start by taking $v = k_n$ in (4.49), whose resulting equation takes us, after we use (2.11) or (2.12), (2.22), (3.14)-(3.15), (5.2) and (5.8), the last just in the case of $P(\mathbf{u}, k) = \varpi(\mathbf{u})k$ a.e. in Ω, together with the inequalities of Hölder and Sobolev, to

$$\int_\Omega \varepsilon_n(k_n)k_n\, d\mathbf{x} \le C, \tag{5.19}$$

where C is a positive constant that does not depend on n. Now, let us prove that $\{\varepsilon_n(k_n)\}_{n\in\mathbb{N}}$ is uniformly integrable. For that, we consider an arbitrary $\epsilon > 0$ and an arbitrary measurable subset $E \subset \Omega$ such that $\mathcal{L}_n(E) < \delta$ for some $\delta > 0$. On the other hand, we can see that for $|k_n| \ge M$, for an arbitrary $M > 0$, we can use (5.19) together with the fact that $\varepsilon_n(k_n)\, k_n \ge 0$ for all k_n (see (4.7)), to show that

$$\int_E |\varepsilon_n(k_n)|\,|k_n|\, d\mathbf{x} \le C \Rightarrow \int_E |\varepsilon_n(k_n)|\, d\mathbf{x} \le \frac{C}{M}. \tag{5.20}$$

As a consequence of (5.19)-(5.20) and of the assumption (3.3), we can write

$$\begin{aligned}\int_E |\varepsilon_n(k_n)|\, d\mathbf{x} &= \int_{E\cap\{|k_n<M\}} |\varepsilon_n(k_n)|\, d\mathbf{x} + \int_{E\cap\{|k_n|\ge M\}} |\varepsilon_n(k_n)|\, d\mathbf{x} \\ &\le \int_{E\cap\{|k_n|<M\}} |\varepsilon_n(k_n)|\, d\mathbf{x} + \frac{C}{M} \le \int_E G_M\, d\mathbf{x} + \frac{C}{M}\end{aligned}$$

for any $M > 0$ and for some positive constant C not depending on n. Next, we take M such that $\frac{C}{M} < \frac{\epsilon}{2}$ and, by the assumption that $G_M \in \mathrm{L}^1(\Omega)$ (see (3.3)), we may choose δ in such a way that

$$\int_E G_M\, d\mathbf{x} \le \|G_M\|_{\mathrm{L}^1(\Omega)}\mathcal{L}_n(E) < \delta\|G_M\|_{\mathrm{L}^1(\Omega)} < \frac{\epsilon}{2}\,.$$

Thus, we have

$$\int_E |\varepsilon_n(k_n)|\, d\mathbf{x} < \epsilon,$$

and, by Vitali's theorem, $\varepsilon(k) \in \mathrm{L}^1(\Omega)$ and

$$\varepsilon_n(k_n) \to \varepsilon(k) \quad \text{strongly in } \mathrm{L}^1(\Omega), \quad \text{as } n \to \infty. \tag{5.21}$$

Now, since the test functions $\varphi \in \mathrm{W}^{1,q'}(\Omega) \hookrightarrow \mathrm{L}^\infty(\Omega)$, it follows that (5.18) is a consequence of (5.21).

As in the precedent section, from (5.10) and (5.21), we have (up to some subsequences)

$$\varepsilon_n(k_n)k_n \to \varepsilon(k)k \quad \text{a.e. in } \Omega, \quad \text{as } n \to \infty. \tag{5.22}$$

Then, due to (5.19) and (5.22), Fatou's lemma yields that $\varepsilon(k)k \in \mathrm{L}^1(\Omega)$.

Finally, we can pass to the limit $n \to \infty$ in the equations (4.48)-(4.49) to obtain (2.5)-(2.6) for any $(\mathbf{v}, \varphi) \in \mathbf{V}\cap\mathrm{L}^\infty(\Omega)\times \mathrm{W}_0^{1,q'}(\Omega)$.

5.4. **To conclude that $k \ge 0$ and $\varepsilon(k) \ge 0$.** This is an immediate consequence of the Subsection 4.4. In fact, using (5.9) together with Sobolev's inequality and (4.54), we obtain

$$\|k^-\|_{\mathrm{L}^q(\Omega)} \le \liminf_{n\to\infty} \|k_n^-\|_{\mathrm{L}^q(\Omega)} \le C \liminf_{n\to\infty} \|\boldsymbol{\nabla} k_n^-\|_{\mathrm{L}^2(\Omega)} \le 0.$$

This implies that $k \ge 0$ a.e. in Ω and, as a consequence of this and of the assumption (2.17), we also end up with $\varepsilon(k) \ge 0$ a.e. in Ω.

The proof of Theorem 3.1 is thus finally finished. □

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