A STATIONARY ONE-EQUATION TURBULENT MODEL WITH APPLICATIONS IN POROUS MEDIA

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Abstract. A one-equation turbulent model is studied in this work in the steady-state and with homogeneous Dirichlet boundary conditions. The considered problem generalizes two distinct approaches that are being used with success in the applications to model different flows through porous media. The novelty of the problem relies on the consideration of the classical Navier-Stokes equations with a feedback forces field, whose presence in the momentum equation will affect the equation for the turbulent kinetic energy (TKE) with a new term that is known as the production and represents the rate at which TKE is transferred from the mean flow to the turbulence. By assuming suitable growth conditions on the feedback forces field and on the function that describes the rate of dissipation of the TKE, as well as on the production term, we will prove the existence of the velocity field and of the TKE. The proof of their uniqueness is made by assuming monotonicity conditions on the feedback forces field and on the turbulent dissipation function, together with a condition of Lipschitz continuity on the production term. The existence of a unique pressure, will follow by the application of a standard version of de Rham's lemma.

Key words and phrases: turbulence, $k-$epsilon modelling, porous media, existence, uniqueness.

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1. Introduction

Turbulent fluid flows through porous media are of considerable theoretical and practical interest in various engineering subjects such as chemical, mechanical, geological or environmental. In the literature of these disciplines, there are two distinct approaches for developing turbulence models based on the averaging theory. The main idea of this theory, relies on the application of two averaging operators to the set of equations that govern the considered problem. These operators, are a volume averaging operator, which is characteristic from the usual derivation of the macroscopic equations of porous media flows, and a Reynolds averaging operator (see *e.g.* [32]) that is necessary to derive the Reynolds averaged equations of turbulent flows. Volume averaging in a porous medium makes use of the concept of a representative elementary volume over which local, or microscopic, equations are integrated, giving rise to the macroscopic governing equations (see e.g. [20]). In a similar fashion, Reynolds averaging of turbulent flows leads to time-mean properties, which are usually described via statistical analysis (see e.g. [9, 26]). For the first approach, the turbulent transport equations are derived by volume averaging the Reynolds-averaged microscopic equation. The additional terms representing production and dissipation of the turbulent kinetic energy, were modeled in [28] by introducing two unknown model constants, which in turn were determined from numerical simulations using a spatially periodic array of square rods. As for the second approach, the turbulent transport equations are derived by time averaging the extended Darcy-Forchheimer model obtained in [37] by volume-averaging the microscopic equations. In [2] the authors made use of the classical eddy diffusivity concept for closure, but they concluded that turbulent models derived directly from the general macroscopic equations do not accurately characterize turbulence induced by the porous matrix. In both developments, the porous medium is considered to be rigid, fixed, isotropic and saturated by an incompressible fluid, and both techniques aim to derive suitable macroscopic transport equations. Still related

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with the first approach, is the methodology developed in [34] where the authors have used a double-decomposition technique to derive the turbulent transport equations. However, to derive the equation for the turbulent kinetic energy these authors proposed a macroscopic Boussinesq assumption and introduced the concepts of macroscopic turbulent viscosity and macroscopic turbulent diffusion. The main feature of the work [34] is that the total drag due to the porous matrix is included in the mean flow equation only after all the other equations are obtained. To account for the porous structure, an additional term was included in the source terms for the turbulent kinetic energy and for the turbulent dissipation equations. To determine the constants for the extra terms (closure of the macroscopic model) the equations for the microscopic flow were numerically solved inside a periodic cell. The authors of [34] concluded in their subsequent studies that time averaging the volume-averaged equations and volume averaging the time-averaged equations lead to similar equations for the momentum equation. However, they pointed out a difference between the expressions used to present the equation for the turbulent kinetic energy (see [23]). Since models based on the first approach do not accurately take into account the turbulence inside the pores, its real applications are scarce and therefore we will not consider them in our mathematical study. In turn, both models [28, 34] have been experimentally tested with success by many authors (see *e.g.* [10, 19, 31]). The model [34] has given good predictions of the turbulent kinetic energy in reacting flows through porous media [31] and in turbulent flows through porous media of particular morphologies [10], whereas the model [28] has shown ability to account for the mixing and mass transfer measure within a packed bed [19].

Motivated by the turbulent models [28, 34], we study, in this work, a one-equation turbulent model for the description of incompressible fluids within a fluid-saturated and rigid porous medium, which for simplicity is also assumed to be fixed and isotropic. The problem is assumed to be governed by the following general set of equations in the steady-state,

$$\mathbf{div}\,\overline{\mathbf{u}} = 0, \tag{1.1}$$

$$(\overline{\mathbf{u}}\cdot\boldsymbol{\nabla})\overline{\mathbf{u}} = \overline{\mathbf{g}} + \mathbf{F}(\overline{\mathbf{u}}) - \boldsymbol{\nabla}\overline{p} + \mathbf{div}\left((\nu + \nu_T(k))\,\mathbf{D}(\overline{\mathbf{u}})\right), \tag{1.2}$$

$$\overline{\mathbf{u}}\cdot\boldsymbol{\nabla}k = \mathrm{div}\,(\nu_D(k)\boldsymbol{\nabla}k) + \nu_T(k)|\mathbf{D}(\overline{\mathbf{u}})|^2 + P(\overline{\mathbf{u}},k) - \varepsilon(k), \tag{1.3}$$

where the velocity field $\overline{\mathbf{u}}$, the pressure $\overline{p}$ and the external forces field $\overline{\mathbf{g}}$ are, in fact, averages that result by the application of two different averaging concepts [28, 34]. The averaged tensor $\mathbf{D}(\overline{\mathbf{u}})$ is the symmetric part of the averaged gradient $\boldsymbol{\nabla}\overline{\mathbf{u}}$, the positive constant ν is the kinematic viscosity and expresses the ratio of the internal forces in the fluid, called dynamic viscosity, to the mass density ρ, assumed to be constant and positive. The function k is an unknown of the problem that was introduced in turbulence modelling by Kolmogorov (see *e.g.* [9, 26]) to characterize the energy of the turbulence in the flow, and therefore it is usually called turbulent kinetic energy (TKE). Turbulent kinetic energy can be produced by fluid shear, friction or buoyancy, or through external forcing at low-frequency eddy scales. Turbulence kinetic energy is then transferred down the turbulence energy cascade, and is dissipated by viscous forces at the Kolmogorov scale. The rate of dissipation of the TKE is described, in the model, by the function ε, which, accordingly, is denoted by dissipation of the TKE, or, briefly, turbulent dissipation. The scalar function ν_T is the (Boussinesq) turbulent viscosity, or eddy viscosity, that, according to Prandtl's hypothesis (see *e.g.* [9]), may depend on k and on ε, whereas ν_D is the turbulent diffusion, or eddy diffusion, that, according to the hypothesis that convection by random fields produces diffusion for the mean flow (see *e.g.* [26]), may also depend on k and on ε. The emergence of the quantity ε in the model, would led us to derive an equation for the transport of this function in order to close the model. However, the consideration of one-equation models, that we assume in this work, is acceptable in the sense that the equation for ε may be discarded by prescribing an appropriate length scale. Consequently the turbulent viscosity ν_T and the turbulent diffusion ν_D are assumed to depend only on k, and, due to Prandtl's hypothesis, the turbulent dissipation ε depends only on k, being considered, in most studies, the Launder-Spalding hypothesis, *i.e.* that ε is of the order of $k^{\frac{3}{2}}$.

In the scope of porous media, all the terms in the momentum equation (1.2) should come affected by the porosity of the medium, say ϕ, which is obtained by applying spatial averaging to the characteristic function of the fluid phase, and therefore may depend on the space variable, and ranging in the interval $(0,1)$ (see *e.g.* [20]). Similar to what we have assumed that other properties of the fluid are constant, such as kinematic viscosity and density, in this work we assume the porosity is also constant. In particular, the feedback term $\mathbf{F}(\overline{\mathbf{u}})$, that characterizes the resistance made by the rigid matrix of the porous medium to the flow, is usually characterized by Darcy's law,

$$\mathbf{F}^D(\overline{\mathbf{u}}) = -c_{Da}\overline{\mathbf{u}}, \quad c_{Da} := \frac{\nu\,\phi}{K}, \tag{1.4}$$

where K is a positive constant accounting for the permeability of the medium. However, as Reynolds number increases, small-scale drag effects, due to the flow through the porous medium, can be captured by adding extra terms to Darcy's law (1.4), giving rise to various non-Darcy models such as the Darcy-Forchheimer,

$$\mathbf{F}^{DF}(\overline{\mathbf{u}}) = -c_{Da}\overline{\mathbf{u}} - c_F\,|\overline{\mathbf{u}}|\overline{\mathbf{u}}, \quad c_F := \frac{C_F\,\phi}{\sqrt{K}}, \tag{1.5}$$

or the Darcy-Forchheimer's power-law,

$$\mathbf{F}^{DF}(\overline{\mathbf{u}}) = -c_{Da}\overline{\mathbf{u}} - c_F|\overline{\mathbf{u}}|^{m-1}\overline{\mathbf{u}}, \tag{1.6}$$

where C_F is the Forchheimer coefficient, a positive constant that is experimentally determined, and $m \geq 1$ is a real constant that characterizes the flow. The additional term $P(\overline{\mathbf{u}}, k)$ in equation (1.3), that appears as an output of the averaging process, is a production term of turbulent kinetic energy and gives account of the solids inside the fluid. As we observed above, the influence of this term in the turbulence equations is distinct for each model [28, 34]. In fact, we have

$$P(\overline{\mathbf{u}}, k) = P(\overline{\mathbf{u}}) := C_{NK}|\overline{\mathbf{u}}|^3, \quad C_{NK} := \frac{39\phi^2(1-\phi)^{\frac{5}{2}}}{D}, \quad \text{for the model [28]} \tag{1.7}$$

$$P(\overline{\mathbf{u}}, k) := C_{PL}|\overline{\mathbf{u}}|\,k, \quad C_{PL} := \frac{0.28}{\sqrt{K}},\ K = \frac{\phi^3 D^2}{144(1-\phi)^2}, \quad \text{for the model [34],} \tag{1.8}$$

where D is the hydraulic diameter, a representative (microscopic) length characterizing the void space of the porous medium.

The influence of feedback forces fields, such as $\mathbf{F}(\mathbf{u})$, in turbulent flows, has been investigated in other fields of applications of the turbulent $k-$epsilon model. Probably the best known situation, happens for fluid flows in a rotating frame, where the Coriolis force

$$\mathbf{F}^C(\mathbf{u}) = -2\Omega \times \mathbf{u}, \quad \Omega \text{ is here the vector of angular velocity,} \tag{1.9}$$

must be considered (see *e.g.* [3]). Another example, is the Lorentz force

$$\mathbf{F}^L(\mathbf{u}, \mathbf{B}) = \mathbf{J} \times \mathbf{B}, \quad \mathbf{J} = \sigma(-\nabla\Phi + \mathbf{u} \times \mathbf{B}), \tag{1.10}$$

a term where the Navier-Stokes equations are coupled to Maxwell's equations and Ohm's law for fluid flows controlled by a magnetic field $\mathbf{B}$. In (1.10), $\mathbf{J}$ is the total electric current intensity, Φ is the electric potential, which is given by the Poisson equation $\Delta\Phi = \mathbf{div}(\mathbf{u} \times \mathbf{B})$, and σ is the conductivity, a material dependent parameter. The effects of the Coriolis force (1.9) in the turbulent equations (1.1)-(1.3), can be easily seen that are

$$\mathbf{F}^C(\overline{\mathbf{u}}) = -2\Omega \times \overline{\mathbf{u}} \quad \text{and} \quad P(\overline{\mathbf{u}}, k) = 0\,. \tag{1.11}$$

However, for the Lorentz force, the Reynolds-averaged analysis does not follow straightforward, because specific interactions of this term in the turbulence equations (1.1)-(1.3) need to be taken into account. In [21] it was numerically studied the effects of the Lorentz force (1.10) in turbulence closure models. It was shown that, in addition to the direct interaction with the mean velocity through the electromagnetic force $\mathbf{F}^L(\mathbf{u}, \mathbf{B})$, which will itself influence the turbulence through the deformed mean rate of strain, the magnetic field affects also the velocity fluctuations by the fluctuating Lorentz force.

Due to the mathematical interest, we consider the problem (1.1)-(1.3) in a general dimension $d \geq 2$, although the dimensions of physics interest are $d = 2$ or $d = 3$. As a consequence of considering a general dimension, we assume the equations (1.1)-(1.3) are satisfied in a bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, with its boundary denoted by $\partial\Omega$. For the sake of simplifying the notation, we shall consider the problem (1.1)-(1.3) written without the bars over the averaged quantities,

$$\mathbf{div}\,\mathbf{u} = 0 \quad \text{in } \Omega, \tag{1.12}$$

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{g} - \mathbf{f}(\mathbf{u}) - \nabla p + \mathbf{div}\left((\nu + \nu_T(k))\,\mathbf{D}(\mathbf{u})\right) \quad \text{in } \Omega, \tag{1.13}$$

$$\mathbf{u}\cdot\nabla k = \operatorname{div}\left(\nu_D(k)\nabla k\right) + \nu_T(k)|\mathbf{D}(\mathbf{u})|^2 + P(\mathbf{u}, k) - \varepsilon(k) \quad \text{in } \Omega\,. \tag{1.14}$$

Note that now in the mean flow equation (1.13), we have highlighted the negative sign of the feedback forces field, by writing $\mathbf{F}(\mathbf{u}) = -\mathbf{f}(\mathbf{u})$, to account for the real situation of applications we are interested in: Coriolis force (1.9), Darcy's drag (1.4) and Darcy-Forchheimer's drag (1.5) or (1.6). We supplement the equations (1.12)-(1.14) with Dirichlet homogeneous boundary conditions,

$$\mathbf{u} = \mathbf{0} \quad \text{and} \quad k = 0 \quad \text{on } \partial\Omega. \tag{1.15}$$

For the sake of a mathematical generalization, we assume that

$$\mathbf{f} : \Omega \times \mathbb{R}^d \longrightarrow \mathbb{R}^d, \quad \varepsilon,\ \nu_T,\ \nu_D : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}, \quad P : \Omega \times \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}.$$

Observe that we are considering the possibility that all the functions $\mathbf{f}$, P, ε, ν_T and ν_D may also depend on the space variable.

The analysis of the problem (1.12)-(1.15) is, to our best knowledge, new in the mathematical literature of the $k-$epsilon turbulent model, though some insights were given in our conference paper [32]. As for the same problem, but in the absence of feedback forces fields, it is being investigated during the last 20 years, although important questions, as the 3-d transient problem, or the case of real turbulent viscosity and turbulent diffusion functions, remain open. For questions of existence, uniqueness and regularity of the solutions, related to the problem with no feedback forces, we address the reader to the works [8, 24] (see also [13], [17] and [12, 14, 18, 22, 30]). With respect to the mathematical derivation of the $k-$epsilon turbulent model, we address the reader to the monographs [9, 26] in the general case, and to [23] for turbulent flows through porous media. The consideration of feedback forces fields have already been studied in [4, 5, 6] to control the stopping distance of laminar fluid flows.

The notation used throughout this article, and the main notions of the considered function spaces, are largely standard in the literature of Partial Differential Equations and in Mathematical Fluid Mechanics as well. We address the reader to the monographs [15, 16, 25, 36] for any question related to that matter.

The introduction to our work is made in the current Section 1, and the rest of the article is organized as follows. In Section 2, we introduce the notion of solutions to the problem (1.12)-(1.15) we are interested in and there we will state the main conditions that both the feedback functions $\mathbf{f}(\mathbf{u})$, $\varepsilon(k)$, $P(\mathbf{u}, k)$ and the turbulent functions $\nu_T(k)$, $\nu_D(k)$ must fulfil. The existence of the velocity field $\mathbf{u}$ and of the turbulent kinetic energy k, will be addressed in Section 3, and its proof will be carried out from that section until Section 5. Under extra higher integrability conditions on the gradient solutions $\boldsymbol{\nabla}\mathbf{u}$ and $\boldsymbol{\nabla}k$, it will be established the uniqueness of $\mathbf{u}$ and k at Section 6 . The existence of a unique pressure p is proved in Section 7.

2. Weak formulation

In order to define the notion of a weak solution to the problem (1.12)-(1.15), let us introduce the following function spaces largely used in the mathematical analysis of fluid problems,

$$\mathcal{V} := \{\mathbf{v} \in \mathbf{C}_0^\infty(\Omega) : \mathbf{div}\,\mathbf{v} = 0\},$$
$$\mathbf{H} := \text{closure of } \mathcal{V} \text{ in } \mathbf{L}^2(\Omega),$$
$$\mathbf{V} := \text{closure of } \mathcal{V} \text{ in } \mathbf{H}^1(\Omega).$$

By $\mathbf{V}'$ we shall denote the dual of the space $\mathbf{V}$. Let us also define the scalar function space

$$\mathrm{V} := \text{closure of } \mathrm{C}_0^\infty(\Omega) \text{ in } \mathrm{H}^1(\Omega).$$

In the mathematical treatment of the turbulence problem (1.12)-(1.15), there is a set of usual assumptions that although do not follow from the real situation they are physically admissible,

$$\mathbf{f} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d \quad \text{is a Carathéodory function,} \tag{2.1}$$

$$\varepsilon,\ \nu_T,\ \nu_D : \Omega \times \mathbb{R} \to \mathbb{R} \quad \text{are Carathéodory functions,} \tag{2.2}$$

$$P : \Omega \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \quad \text{is a Carathéodory function.} \tag{2.3}$$

Observe again that, in view of these assumptions, all these functions may also depend on the space variable. In particular, the assumption (2.2) fits with turbulent dissipation, turbulent viscosity and turbulent diffusion functions involved in realistic models when giving, for instance, by the following formulae

$$\varepsilon(\mathbf{x}, k) = \frac{k\sqrt{k}}{l(\mathbf{x})}, \qquad \nu_T(\mathbf{x}, k) = C_1 l(\mathbf{x})\sqrt{k}, \qquad \nu_D(\mathbf{x}, k) = \mu_e + C_2 l(\mathbf{x})\sqrt{k}, \qquad l \neq 0, \quad k \geq 0, \tag{2.4}$$

where μ_e is an effective (dynamic) viscosity, C_1, C_2 are dimensionless constants and $l : \Omega \to \mathbb{R}$ is the mixing length function which is usually assumed to satisfy $l(\mathbf{x}) \geq l_0$ for a.e. $\mathbf{x} \in \Omega$ and for some positive constant l_0 (see *e.g.* [9, 23, 26, 30]).

There is another set of assumptions that impose some restrictions on the physics of the problem, but are mathematically needed. We assume the boundedness of both turbulent viscosity and turbulent diffusion,

$$|\nu_T(\mathbf{x}, k)| \leq C_T, \quad |\nu_D(\mathbf{x}, k)| \leq C_D \quad \text{a.e. in } \Omega, \tag{2.5}$$

for some positive constants C_T and C_D. The question of whether we can remove the upper bounds of the turbulent viscosity and of the turbulent diffusion is an open problem, at least in the function spaces setting we are going to use, for the problem with convection and in the case of non periodic boundary conditions. See, for instance [12, 14, 18, 22, 30], for the turbulent $k-$epsilon model in the absence of feedback forces fields. Below we give a notion of weak solution to the problem (1.12)-(1.15) as comprehensive as possible.

Definition 2.1. *Let the conditions (2.1)-(2.5) be fulfilled and assume that* $\mathbf{g} \in \mathbf{V}'$. *We say a pair* $(\mathbf{u}, k)$ *is a weak solution to the problem (1.12)-(1.15), if:*

(1) $\mathbf{u} \in \mathbf{V}$ *and for every* $\mathbf{v} \in \mathbf{V} \cap \mathbf{L}^d(\Omega)$ *there hold* $\mathbf{f}(\mathbf{x}, \mathbf{u}) \cdot \mathbf{v} \in \mathbf{L}^1(\Omega)$ *and*

$$\int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v}\, d\mathbf{x} + \int_\Omega (\nu + \nu_T(\mathbf{x}, k))\, \mathbf{D}(\mathbf{u}) : \nabla \mathbf{v}\, d\mathbf{x} + \int_\Omega \mathbf{f}(\mathbf{x}, \mathbf{u}) \cdot \mathbf{v}\, d\mathbf{x} = \int_\Omega \mathbf{g} \cdot \mathbf{v}\, d\mathbf{x}; \tag{2.6}$$

(2) $k \in \mathrm{W}_0^{1,q}(\Omega)$, *with* $\frac{2d}{d+2} \leq q < d'$, *and for every* $\varphi \in \mathrm{W}_0^{1,q'}(\Omega)$ *there hold* $\varepsilon(\mathbf{x}, k)\, \varphi$, $P(\mathbf{x}, \mathbf{u}, k)\, \varphi \in \mathrm{L}^1(\Omega)$ *and*

$$\int_\Omega (\mathbf{u} \cdot \nabla k)\varphi\, d\mathbf{x} + \int_\Omega \nu_D(\mathbf{x}, k)\nabla k \cdot \nabla \varphi\, d\mathbf{x} + \int_\Omega \varepsilon(\mathbf{x}, k)\, \varphi\, d\mathbf{x} = \int_\Omega \nu_T(\mathbf{x}, k)|\mathbf{D}(\mathbf{u})|^2 \varphi\, d\mathbf{x} + \int_\Omega P(\mathbf{x}, \mathbf{u}, k)\, \varphi\, d\mathbf{x}; \tag{2.7}$$

(3) $k \geq 0$ *and* $\varepsilon(\mathbf{x}, k) \geq 0$ a.e. *in* Ω.

Remark 2.1. *To simplify the exposition, we assume, in the rest of the work, that the general space dimension d satisfies to*

$$2 \leq d \leq 4.$$

In this case, the Sobolev imbedding $\mathbf{H}_0^1(\Omega) \hookrightarrow \mathbf{L}^d(\Omega)$ *holds and therefore it is only needed to require the test functions of (2.6) are in the function space* $\mathbf{V}$.

In this work, we consider the case when we impose restrictions on the growths of the feedback force $\mathbf{f}(\mathbf{x}, \mathbf{u})$, of the turbulent dissipation function $\varepsilon(\mathbf{x}, k)$ and of the production term $P(\mathbf{x}, \mathbf{u}, k)$.

We assume the existence of nonnegative constants C_f and C_ε such that

$$|\mathbf{f}(\mathbf{x}, \mathbf{u})| \leq C_f |\mathbf{u}|^\alpha \quad \text{for} \quad 0 \leq \alpha \leq \frac{d+2}{d-2} \text{ if } d \neq 2, \quad \text{or for any } \alpha \geq 0 \text{ if } d = 2, \tag{2.8}$$

$$|\varepsilon(\mathbf{x}, k)| \leq C_\varepsilon |k|^\theta \quad \text{for} \quad 0 \leq \theta < \frac{d}{d-2} \text{ if } d \neq 2, \quad \text{or for any } \theta \geq 0 \text{ if } d = 2, \tag{2.9}$$

for a.a. $\mathbf{x} \in \Omega$. Observe that the situations of considering a Coriolis force or a Darcy or Darcy-Forchheimer drag's force correspond to the case of assumption (2.8) with $\alpha = 1$ or $\alpha = 2$. In contrast, the limit cases $\alpha = 0$ or $C_f = 0$, correspond to the situation of considering a one-equation turbulent model not affected by external feedback forces. On the other hand, assumption (2.9) covers the Launder-Spalding expression for the turbulent dissipation when $\theta = \frac{3}{2}$. The case of $\theta = 0$ and $C_\varepsilon > 0$ in (2.9), correspond to the position when the rate of dissipation of the TKE is constant, whereas the case $C_\varepsilon = 0$ accounts for a theoretical situation with no turbulent dissipation at all.

On the production term $P(\mathbf{x}, \mathbf{u}, k)$, we shall assume the possibilities arising in the applications aforementioned. We thus consider the cases of

$$P(\mathbf{x}, \mathbf{u}, k) = \pi(\mathbf{x}, \mathbf{u}) \quad \text{or} \quad P(\mathbf{x}, \mathbf{u}, k) = \varpi(\mathbf{x}, \mathbf{u})k, \tag{2.10}$$

where, accordingly to (2.3),

$$\pi,\ \varpi : \Omega \times \mathbb{R}^d \to \mathbb{R} \quad \text{are Carathéodory functions.} \tag{2.11}$$

But then, in order to be physically realistic with the turbulent models [28, 34], we need to assume that

$$\pi(\mathbf{x}, \mathbf{u}) \geq 0 \quad \text{and} \quad \varpi(\mathbf{x}, \mathbf{u}) \geq 0 \quad \text{for all } \mathbf{u} \in \mathbb{R}^d \text{ and for a.a } \mathbf{x} \in \Omega. \tag{2.12}$$

If $P(\mathbf{x},\mathbf{u},k)=\pi(\mathbf{x},\mathbf{u})$, we assume the existence of a nonnegative constant C_π such that

$$|\pi(\mathbf{x},\mathbf{u})|\leq C_\pi|\mathbf{u}|^\beta \quad \text{for} \quad 0\leq\beta\leq\frac{d+2}{d-2} \text{ if } d\neq 2, \quad \text{or for} \quad \text{any } \beta\geq 0 \text{ if } d=2, \tag{2.13}$$

for a.a. $\mathbf{x}\in\Omega$. The influence of the Darcy-Forchheimer drag's force (1.5) in the turbulent model [28] of porous media flows, is accounted by assumption (2.13) when we take there $\beta=3$ and $C_\pi=C_{NK}$, being C_{NK} given at (1.7). Note that the case of $P(\mathbf{x},\mathbf{u},k)=0$ corresponds to take $C_\pi=0$ in (2.13) and it is a situation when there is no production term in the transport equation for the TKE, and that accounts, for instance, for the influence of the Coriolis force in a turbulent $k-$epsilon model.

If $P(\mathbf{x},\mathbf{u},k)=\varpi(\mathbf{x},\mathbf{u})k$, we assume the existence of a positive constant C_ϖ such that

$$|\varpi(\mathbf{x},\mathbf{u})|\leq C_\varpi|\mathbf{u}|^\beta \quad \text{for} \quad 0\leq\beta<\frac{4}{d-2} \text{ if } d\neq 2, \quad \text{or for any } \beta\geq 0 \text{ if } d=2, \tag{2.14}$$

for a.a. $\mathbf{x}\in\Omega$. This assumption covers the case of the influence of the Darcy-Forchheimer drag's force (1.5) in the turbulent model [34] of porous media flows when we take $\beta=1$ and $C_\varpi=C_{PL}$, where C_{PL} is given at (1.8). The limit case of $\beta=0$ in the assumption (2.14), matches with the occurrence of a production term $P(\mathbf{x},\mathbf{u},k)$ that depends only on k and might be characteristic of turbulent porous media flows under the influence of Darcy drag's force alone. Observe that, under the assumption that (2.8)-(2.9), (2.10) and (2.13), or (2.14), hold, the conditions of the Definition 2.1 that $\mathbf{f}(\mathbf{x},\mathbf{u})\cdot\mathbf{v}\in\mathbf{L}^1(\Omega)$ and $\varepsilon(\mathbf{x},k)\,\varphi$, $P(\mathbf{x},\mathbf{u},k)\,\varphi\in\mathrm{L}^1(\Omega)$, are easily satisfied.

Remark 2.2. *Growth conditions (2.8), (2.9) and (2.13) or (2.14) can be substituted by these more general growth conditions, respectively:*

$$|\mathbf{f}(\mathbf{x},\mathbf{u})|\leq C_f(1+|\mathbf{u}|)^\alpha \quad \textit{for} \quad 0\leq\alpha\leq\frac{d+2}{d-2} \textit{ if } d\neq 2, \quad \textit{or for any } \alpha\geq 0 \textit{ if } d=2;$$
$$|\varepsilon(\mathbf{x},k)|\leq C_\varepsilon(1+|k|)^\theta \quad \textit{for} \quad 0\leq\theta<\frac{d}{d-2} \textit{ if } d\neq 2, \quad \textit{or for any } \theta\geq 0 \textit{ if } d=2;$$
$$|\pi(\mathbf{x},\mathbf{u})|\leq C_\pi(1+|\mathbf{u}|)^\beta \quad \textit{for} \quad 0\leq\beta\leq\frac{d+2}{d-2} \textit{ if } d\neq 2, \quad \textit{or for any } \beta\geq 0 \textit{ if } d=2;$$
$$|\varpi(\mathbf{x},\mathbf{u})|\leq C_\varpi(1+|\mathbf{u}|)^\beta \quad \textit{for} \quad 0\leq\beta<\frac{4}{d-2} \textit{ if } d\neq 2, \quad \textit{or for any } \beta\geq 0 \textit{ if } d=2;$$

for a.a. $\mathbf{x}\in\Omega$.

Additionally to the aforementioned growth conditions, we assume the following sign conditions,

$$\mathbf{f}(\mathbf{x},\mathbf{u})\cdot\mathbf{u}\geq 0 \quad \text{for all } \mathbf{u}\in\mathbb{R}^d \text{ and for a.a. } \mathbf{x}\in\Omega, \tag{2.15}$$
$$\varepsilon(\mathbf{x},k)\,k\geq 0 \quad \text{for all } k\in\mathbb{R} \text{ and for a.a. } \mathbf{x}\in\Omega\,. \tag{2.16}$$

Condition (2.15) is satisfied by any feedback forces field that we have considered above: Coriolis (1.9), Darcy (1.4) and Darcy-Forchheimer (1.5) or (1.6). With respect to (2.16), this condition is always verified due to the definition of k and $\varepsilon(\mathbf{x},k)$ in the turbulent $k-$epsilon model. In fact, by its definition in the physical situation, $k\geq 0$, and the best known expression for the turbulent dissipation (in one-equation turbulent models) is given by Prandtl's formula $(2.4)_1$. Motivated by this expression, we consider, in this work, that our general turbulent dissipation function can be written in such a way that

$$\varepsilon(\mathbf{x},k)=ke(\mathbf{x},k) \qquad \text{where} \quad e:\Omega\times\mathbb{R}\to\mathbb{R}_0 \text{ is a Carathéodory function.} \tag{2.17}$$

Gathering the information of (2.16) and (2.17) it follows immediately that

$$e(\mathbf{x},k)\geq 0 \quad \text{for all } k\in\mathbb{R} \text{ and for a.a. } \mathbf{x}\in\Omega\,. \tag{2.18}$$

Note that in the particular case of the Prandtl formula $(2.4)_1$, the function $e(\mathbf{x},k)=\frac{\sqrt{k}}{l(\mathbf{x})}$ satisfies to (2.18) whenever $k\geq 0$ and $l\geq l_0$ a.e. in Ω and for some positive constant l_0.

There is another set of assumptions, already touched on at (2.5), that are mathematically needed,

$$0\leq\nu_T(\mathbf{x},k)\leq C_T \quad \text{for all } k\in\mathbb{R} \text{ and for a.a. } \mathbf{x}\in\Omega, \quad C_T\in\mathbb{R}^+, \tag{2.19}$$
$$0<c_D\leq\nu_D(\mathbf{x},k)\leq C_D \quad \text{for all } k\in\mathbb{R} \text{ and for a.a. } \mathbf{x}\in\Omega, \quad c_D,C_D\in\mathbb{R}^+. \tag{2.20}$$

To avoid the trivial solution $k = 0$, we shall assume in the sequel, and in addition to (2.19), that

$$\nu_T(\mathbf{x}, k) \neq 0 \quad \text{when } k = 0\,. \tag{2.21}$$

Due to the presence of the production term $P(\mathbf{x}, \mathbf{u}, k)$ (with a positive sign) on the right-hand side of the equation (1.14), we have to impose a suitable restriction related with the best constant of the Sobolev inequality. We start by recalling that the principal (positive) eigenvalue, say $\lambda_P(d)$, for the Laplacian problem

$$\begin{cases} \Delta\phi = -\lambda\phi & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega \end{cases} \tag{2.22}$$

can be characterized, by the Rayleigh quotient, in such a way that

$$\frac{1}{\lambda_P(d)} = \inf\left\{\frac{\|\boldsymbol{\nabla}\phi\|^2_{\mathrm{L}^2(\Omega)}}{\|\phi\|^2_{\mathrm{L}^2(\Omega)}} : \phi \in \mathrm{H}^1_0(\Omega),\ \phi \neq 0\right\}.$$

It is well known that $\lambda_P(d)$ is attained and $0 < \lambda_P(d) < \infty$ (see *e.g.* [15]). Moreover, $\lambda_P(d)$ is the best possible constant in the Poincaré's inequality,

$$\|\phi\|^2_{\mathrm{L}^2(\Omega)} \leq \lambda_P(d)\|\boldsymbol{\nabla}\phi\|^2_{\mathrm{L}^2(\Omega)} \quad \forall\, \phi \in \mathrm{H}^1_0(\Omega). \tag{2.23}$$

In order to simplify the exposition in the sequel, we rename the positive constant $\lambda_P(d)$ in (2.23) as $\lambda_P(d)^2$. The extension of (2.23) to a general L^r norm was studied by many authors. In particular, it was proved in [1, 35] that, for $1 \leq r < d$, $d \geq 2$ and $r^* = \frac{dr}{d-r}$,

$$\frac{1}{\lambda(r, d)} = \inf\left\{\frac{\|\boldsymbol{\nabla}\phi\|_{\mathrm{L}^r(\Omega)}}{\|\phi\|_{\mathrm{L}^{r^*}(\Omega)}} : \phi \in \mathrm{W}^{1,r}_0(\Omega),\ \phi \neq 0\right\} \tag{2.24}$$

is achieved and $0 < \lambda(r, d) < \infty$. The critical case of $d = r$ in (2.24) follows from the following Moser's inequality [27], an improvement of earlier results due to Pohozhaev and to Trudinger,

$$\sup_{\|\phi\|_{\mathrm{W}^{1,d}_0(\Omega)} \leq 1} \int_\Omega e^{\alpha|\phi|^{d'}} d\mathbf{x} \begin{cases} \leq C\mathcal{L}^d(\Omega) & \text{if } \alpha \leq \alpha_d \\ = \infty & \text{if } \alpha > \alpha_d, \end{cases}$$

where $\alpha_d = d\omega_{d-1}^{\frac{1}{d-1}}$, where ω_{d-1} denotes here the volume of the unit ball in $\mathbb{R}^{d-1}$, and C is a positive constant depending only on d. In particular, for $d = r = 2$,

$$\frac{1}{\lambda(2, 2)} = \inf\left\{\frac{\|\boldsymbol{\nabla}\phi\|_{\mathrm{L}^2(\Omega)}}{\|\phi\|_{\mathrm{L}^{2^*}(\Omega)}} : \phi \in \mathrm{H}^1_0(\Omega),\ \phi \neq 0\right\} \tag{2.25}$$

is attained and $0 < \lambda(2, 2) < \infty$. As a consequence of (2.24)-(2.25), the sharpest Sobolev constant's inequality $\lambda(2, d)$ is attained in such a way that $0 < \lambda(2, d) < \infty$ and

$$\|\phi\|_{\mathrm{L}^{2^*}(\Omega)} \leq \lambda(2, d)\|\boldsymbol{\nabla}\phi\|_{\mathrm{L}^2(\Omega)} \quad \forall\, \phi \in \mathrm{H}^1_0(\Omega), \quad \forall d \geq 2\,. \tag{2.26}$$

In the sequel we shall denote with a capital Λ the best constants of the vectorial versions of the Sobolev inequalities described by (2.23)-(2.26). Observe that in the vectorial versions, the best constants of the Sobolev inequalities are not necessarily the same as in the scalar forms. But it can be easily shown that, for instance, for the scalar and vectorial versions of (2.26), the best constants of these Sobolev inequalities satisfy to $\Lambda(2, d) \leq d^{\frac{3}{2}+\frac{1}{2^*}}\lambda(2, d)$.

3. Existence

In this section, we state the existence result and we start here with the approach to prove it. For the sake of simplifying the writing, from now on, we shall no longer write the dependence of the Carathéodory functions (2.1)-(2.3), (2.11) and (2.17) on the space variable $\mathbf{x}$. We will split the existence result into two main different cases, $P(\mathbf{u}, k) = \pi(\mathbf{u})$ and $P(\mathbf{u}, k) = \varpi(\mathbf{u})k$, arisen in the applications of the turbulent models [28, 34] in porous media flows we have mentioned in Section 1. It should be remarked that the conditions under which we prove the existence for the case $P(\mathbf{u}, k) = \varpi(\mathbf{u})k$ are more restrictive. In particular for this case, we will require a smallness condition on the problem data in comparison with the kinematic viscosity and with the constant of uniform ellipticity (the lower bound of the turbulent diffusion function). The purpose of the following theorem is twofold in the sense that encompasses the two distinct models [28, 34] for turbulent flows through porous media we consider in this work.

Theorem 3.1. *Let Ω be a bounded domain of $\mathbb{R}^d$, $2 \leq d \leq 4$, with a Lipschitz-continuous boundary $\partial\Omega$. Assume all the conditions (2.1)-(2.3), (2.8)-(2.9), (2.10)-(2.12), (2.15)-(2.16), (2.17) and (2.19)-(2.21) hold. In addition, assume*

$$\mathbf{g} \in \mathbf{L}^2(\Omega), \tag{3.1}$$

and that one of the following conditions is satisfied:

(1) *$P(\mathbf{u}, k) = \pi(\mathbf{u})$ a.e. in Ω and (2.13) holds;*
(2) *$P(\mathbf{u}, k) = \varpi(\mathbf{u})k$ a.e. in Ω, (2.14) holds, and*

$$c_D > C\left(\frac{\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}}{\nu}\right)^{\beta} \tag{3.2}$$

for the positive constant C defined at (4.11), and where ν is the positive constant that accounts for the kinematic viscosity.

Then, there exists, at least, a weak solution to the problem (1.12)-(1.15).

Remark 3.1. *Observe that the possibility of $P(\mathbf{u}, k) = 0$ a.e. in Ω, corresponding to the influence of the Coriolis force (1.9) in the turbulent equations (see (1.11)), is covered by condition (1), in particular when we take $C_\pi = 0$ in (2.13). On the other hand, the case of $P(\mathbf{u}, k) = k$ a.e. in Ω, characteristic of turbulent flows through porous media under the solely influence of Darcy's drag force (1.4), is contained in condition (2) when we take $C_\varpi = 1$ and $\beta = 0$ in (2.14).*

Proof. The proof of Theorem 3.1 is going to be split in the rest of this section and through Sections 4 and 5. To start with the proof, we first observe that, since the third term of (1.14) is only in L^1, we will use a regularization technique (see *e.g.* [29]) to deal with this term. Therefore, we start by considering, for each $n \in \mathbb{N}$, the following regularized problem

$$\mathbf{div}\,\mathbf{u} = 0 \quad \text{in} \quad \Omega, \tag{3.3}$$

$$(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = \mathbf{g} - \mathbf{f}(\mathbf{u}) - \boldsymbol{\nabla} p + \mathbf{div}\left((\nu + \nu_T(k))\,\mathbf{D}(\mathbf{u})\right) \quad \text{in } \Omega, \tag{3.4}$$

$$\mathbf{u} \cdot \boldsymbol{\nabla} k = \operatorname{div}\left(\nu_D(k)\boldsymbol{\nabla} k\right) + \nu_T(k)\,\mathcal{R}_n\Big(|\mathbf{D}(\mathbf{u})|^2\Big) + P(\mathbf{u}, k) - \varepsilon(k) \quad \text{in } \Omega, \tag{3.5}$$

$$\mathbf{u} = 0 \quad \text{and} \quad k = 0 \quad \text{on } \partial\Omega, \tag{3.6}$$

where $\mathcal{R}_n(a)$ denotes the following regularization of the nonnegative term a

$$\mathcal{R}_n(a) := \frac{a}{1 + \frac{1}{n}a}\,. \tag{3.7}$$

Observe that this regularizing term satisfies to

$$\mathcal{R}_n(a) \leq \min\{a,\, n\}\,. \tag{3.8}$$

Under the assumptions of Definition 2.1, we say a pair $(\mathbf{u}, k)$ is a weak solution to the regularized problem (3.3)-(3.6) if, for each $n \in \mathbb{N}$, (1) and (3) of Definition 2.1 hold, and

(2') $k \in \mathrm{H}^1_0(\Omega)$ and for every $\varphi \in \mathrm{H}^1_0(\Omega) \cap \mathrm{L}^d(\Omega)$ there holds

$$\int_\Omega (\mathbf{u} \cdot \boldsymbol{\nabla} k)\varphi\, d\mathbf{x} + \int_\Omega \nu_D(k)\boldsymbol{\nabla} k \cdot \boldsymbol{\nabla}\varphi\, d\mathbf{x} + \int_\Omega \varepsilon(k)\,\varphi\, d\mathbf{x} = \int_\Omega \nu_T(k)\mathcal{R}_n\Big(|\mathbf{D}(\mathbf{u})|^2\Big)\varphi\, d\mathbf{x} + \int_\Omega P(\mathbf{u}, k)\,\varphi\, d\mathbf{x}. \tag{3.9}$$

When $d \leq 4$, it is also only needed to require the test functions φ are in the function space $\mathrm{H}^1_0(\Omega)$, as it was observed in Remark 2.1. The existence of a weak solution to the problem (3.3)-(3.6) is established in the following proposition.

Proposition 3.1. *Let the conditions of Theorem 3.1 be fulfilled. Then (for each $n \in \mathbb{N}$) there exists, at least, a weak solution to the problem (3.3)-(3.6).*

Proof. The proof of Proposition 3.1 is organized in several steps in the next section.

4. Proof of Proposition 3.1

4.1. **Existence of approximate solutions.** Let $\{(\mathbf{v}_i, \upsilon_i)\}_{i=1}^{\infty}$ be a set of non-trivial solutions $(\mathbf{v}_i, \upsilon_i)$, associated to the eigenvalues $\Lambda_i > 0$ and $\lambda_i > 0$, $i = 1, 2, ...$, of the following spectral problems,

$$\begin{cases} \displaystyle\sum_{|\alpha|=s} \int_\Omega D^\alpha \mathbf{v}_i \cdot D^\alpha \mathbf{w}\, d\mathbf{x} = \Lambda_i \int_\Omega \mathbf{v}_i \cdot \mathbf{w}\, d\mathbf{x} \qquad \forall\, \mathbf{w} \in \mathbf{V}, \\ \mathbf{v}_i \in \mathbf{V}, \end{cases}$$

$$\begin{cases} \displaystyle\sum_{|\alpha|=r} \int_\Omega D^\alpha \upsilon_i D^\alpha \omega\, d\mathbf{x} = \lambda_i \int_\Omega \upsilon_i \omega\, d\mathbf{x} \quad \forall\, \omega \in \mathrm{V}, \\ \upsilon_i \in \mathrm{V}, \end{cases}$$

where $\mathbf{V}$ and V are the function spaces introduced at the beginning of Section 2. The family $\{\mathbf{v}_i\}_{i=1}^{\infty}$ is orthogonal in $\mathbf{V}$ and can be chosen as being orthonormal in $\mathbf{H}$ (see *e.g.* [25]), whereas the family $\{\upsilon_i\}_{i=1}^{\infty}$ is orthogonal in V and can be chosen as being orthonormal in $\mathrm{L}^2(\Omega)$ (see *e.g.* [11]). Given $j \in \mathbb{N}$, let us consider the correspondingly j-dimensional spaces $\mathbf{V}^j$ and V^j spanned by $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_j$ and by $\upsilon_1, \upsilon_2, ..., \upsilon_j$, respectively. For each $j \in \mathbb{N}$, we search for an approximate solution $(\mathbf{u}_j, k_j)$ of (2.6) and of (3.9) in the form

$$\mathbf{u}_j = \sum_{i=1}^{j} c_{ij} \mathbf{v}_i, \qquad c_{ij} \in \mathbb{R}, \quad \mathbf{v}_i \in \mathbf{V}^j, \tag{4.1}$$

$$k_j = \sum_{i=1}^{j} d_{ij} \upsilon_i, \qquad d_{ij} \in \mathbb{R}, \quad \upsilon_i \in V^j. \tag{4.2}$$

These functions are found by solving the following system of $2j$ nonlinear algebraic equations, with respect to the $2j$ unknowns $c_{1j},\ c_{2j},\ \dots,\ c_{jj}$ and $d_{1j},\ d_{2j},\ \dots,\ d_{jj}$ obtained from (2.6) and from (3.9), respectively:

$$\begin{aligned} &\int_\Omega ((\mathbf{u}_j \cdot \nabla)\mathbf{u}_j) \cdot \mathbf{v}_i\, d\mathbf{x} + \int_\Omega (\nu + \nu_T(k_j))\mathbf{D}(\mathbf{u}_j) : \nabla \mathbf{v}_i\, d\mathbf{x} + \int_\Omega \mathbf{f}(\mathbf{u}_j) \cdot \mathbf{v}_i\, d\mathbf{x} \\ = &\int_\Omega \mathbf{g} \cdot \mathbf{v}_i\, d\mathbf{x} \quad \text{for } i = 1,\ \dots,\ j; \end{aligned} \tag{4.3}$$

$$\begin{aligned} &\int_\Omega (\mathbf{u}_j \cdot \nabla k_j)\upsilon_i\, d\mathbf{x} + \int_\Omega \nu_D(k_j) \nabla k_j \cdot \nabla \upsilon_i\, d\mathbf{x} + \int_\Omega \varepsilon(k_j)\upsilon_i\, d\mathbf{x} \\ = &\int_\Omega \nu_T(k_j)\, \mathcal{R}_n\Big(|\mathbf{D}(\mathbf{u}_j)|^2\Big)\, \upsilon_i\, d\mathbf{x} + \int_\Omega P(\mathbf{u}_j, k_j)\, \upsilon_i\, d\mathbf{x}, \quad i = 1, \dots, j, \end{aligned} \tag{4.4}$$

Due to the assumptions (2.15)-(2.16), (2.19)-(2.20) and (3.1), we can use a variant of Brower's theorem (see *e.g.* [36, Lemma II.1.4]) to prove the existence of, at least, a solution to the system formed by (4.1)-(4.2) and (4.3)-(4.4). To do it so, we consider a function $\mathcal{P}$, from $\mathbf{V}^j \times V^j$ into itself defined in such a way that

$$\begin{aligned} &\mathcal{P}(\mathbf{v}, \upsilon) \cdot (\mathbf{v}, \upsilon) := \\ &\int_\Omega ((\mathbf{v} \cdot \nabla)\mathbf{v}) \cdot \mathbf{v}\, d\mathbf{x} + \int_\Omega (\nu + \nu_T(\upsilon))\mathbf{D}(\mathbf{v}) : \nabla \mathbf{v}\, d\mathbf{x} + \int_\Omega \mathbf{f}(\mathbf{v}) \cdot \mathbf{v}\, d\mathbf{x} - \int_\Omega \mathbf{g} \cdot \mathbf{v}\, d\mathbf{x} + \\ &\int_\Omega (\mathbf{v} \cdot \nabla \upsilon)\upsilon\, d\mathbf{x} + \int_\Omega \nu_D(\upsilon)|\nabla \upsilon|^2\, d\mathbf{x} + \int_\Omega \varepsilon(\upsilon)\upsilon\, d\mathbf{x} - \int_\Omega \nu_T(\upsilon)\, \mathcal{R}_n\Big(|\mathbf{D}(\mathbf{v})|^2\Big)\, \upsilon\, d\mathbf{x} - \int_\Omega P(\mathbf{v}, \upsilon)\, \upsilon\, d\mathbf{x} \\ &:= I_1 + \cdots - I_4 + \cdots - I_8 - I_9 \end{aligned} \tag{4.5}$$

for all $(\mathbf{v}, \upsilon) \in \mathbf{V}^j \times V^j$ and where the scalar product is induced by $\mathbf{V} \times V$. Evidently, $\mathcal{P}$ so defined is continuous. We first observe that, since $\mathbf{v} \in \mathbf{V}^j$ implies $\mathbf{div}\, \mathbf{v} = 0$, $I_1 = 0$ and $I_5 = 0$. By one hand, due to the assumption (2.19) and to the symmetry of $\mathbf{D}(\mathbf{v})$, and still using Korn's inequality in the term I_2, and on the other hand using (2.20) in I_6, we have

$$I_2 \geq \nu C_K^2 \|\nabla \mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} \quad \text{and} \quad I_6 \geq c_D \|\nabla \upsilon\|^2_{\mathrm{L}^2(\Omega)},$$

where C_K is the Korn inequality's constant. As a consequence of (2.15)-(2.16), $I_3 \geq 0$ and $I_7 \geq 0$. On the other hand, by using Hölder's inequality together with a vectorial version of Poincaré's inequality (2.23),

$$I_4 = \int_\Omega \mathbf{g} \cdot \mathbf{v}\, d\mathbf{x} \leq \Lambda_P(d) \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)} \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\Omega)}.$$

Then, we can see that assumption (2.19), together with (3.7)-(3.8), Hölder's inequality and Sobolev's inequality (2.26), imply

$$I_8 = \int_\Omega \nu_T(\upsilon)\,\mathcal{R}_n\Big(|\mathbf{D}(\mathbf{v})|^2\Big)\,\upsilon\,d\mathbf{x} \leq C_T\, n\|\upsilon\|_{\mathrm{L}^1(\Omega)} \leq C_T\, n\,\sqrt{\mathcal{L}^d(\Omega)}\lambda(2,d)\|\nabla\upsilon\|_{\mathrm{L}^2(\Omega)}\,. \tag{4.6}$$

Finally it remains to estimate properly the term I_9.

4.1.1. ***If*** $P(\mathbf{u},k) = \pi(\mathbf{u})$ ***a.e. in*** Ω, we use the assumption (2.13) together with Hölder's inequality, both scalar and vectorial versions of Sobolev's inequality (2.26), and still Cauchy's inequality, which yield

$$I_9 = \int_\Omega \pi(\mathbf{v})\upsilon\,d\mathbf{x} \leq C_\pi\|\mathbf{v}\|^\beta_{\mathbf{L}^{2^*}(\Omega)}\|\upsilon\|_{\mathrm{L}^{2^*}(\Omega)} \leq C_\pi\,\lambda(2,d)\Lambda(2,d)^\beta\|\nabla\mathbf{v}\|^\beta_{\mathbf{L}^2(\Omega)}\|\nabla\upsilon\|_{\mathrm{L}^2(\Omega)}. \tag{4.7}$$

Then, gathering the information of the estimates of $I_1,\ \dots,\ I_9$, it follows from (4.5) that

$$\begin{aligned}&\mathcal{P}(\mathbf{v},\upsilon)\cdot(\mathbf{v},\upsilon) \geq \\ &\|\nabla\mathbf{v}\|_{\mathbf{L}^2(\Omega)}\left(\nu\, C_K^2\|\nabla\mathbf{v}\|_{\mathbf{L}^2(\Omega)} - \Lambda_P(d)\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}\right) + \\ &\|\nabla\upsilon\|_{\mathrm{L}^2(\Omega)}\left(c_D\|\nabla\upsilon\|_{\mathrm{L}^2(\Omega)} - C_T\, n\,\sqrt{\mathcal{L}^d(\Omega)}\lambda(2,d) - C_\pi\,\lambda(2,d)\Lambda(2,d)^\beta\|\nabla\mathbf{v}\|^\beta_{\mathbf{L}^2(\Omega)}\right).\end{aligned} \tag{4.8}$$

Therefore, $\mathcal{P}(\mathbf{v},\upsilon)\cdot(\mathbf{v},\upsilon) > 0$ for $\|\mathbf{v}\|_{\mathbf{V}} = \rho$ and $\|\upsilon\|_V = \varsigma$, and ρ and ς sufficiently large: more precisely for $\rho > \frac{\Lambda_P(d)}{\nu C_K^2}\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}$ and $\varsigma > \frac{1}{c_D}\left(C_T\, n\,\sqrt{\mathcal{L}^d(\Omega)}\lambda(2,d) + C_\pi\,\lambda(2,d)\Lambda(2,d)^\beta\rho^\beta\right)$.

4.1.2. ***If*** $P(\mathbf{u},k) = \varpi(\mathbf{u})k$ ***a.e. in*** Ω, then we have

$$I_9 = \int_\Omega \varpi(\mathbf{v})\,\upsilon^2\,d\mathbf{x} \leq C_\varpi\|\mathbf{v}\|^\beta_{\mathbf{L}^{2^*}(\Omega)}\|\upsilon\|^2_{\mathrm{L}^{2^*}(\Omega)} \leq C_\varpi\,\lambda(2,d)^2\Lambda(2,d)^\beta\|\nabla\mathbf{v}\|^\beta_{\mathbf{L}^2(\Omega)}\|\nabla\upsilon\|^2_{\mathrm{L}^2(\Omega)}, \tag{4.9}$$

by the application of (2.14) together with Hölder's inequality and both scalar and vectorial versions of Sobolev's inequality (2.26). In this case, the counter part of (4.8) is

$$\begin{aligned}&\mathcal{P}(\mathbf{v},\upsilon)\cdot(\mathbf{v},\upsilon) \geq \\ &\|\nabla\mathbf{v}\|_{\mathbf{L}^2(\Omega)}\left(\nu\, C_K^2\|\nabla\mathbf{v}\|_{\mathbf{L}^2(\Omega)} - \Lambda_P(d)\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}\right) + \\ &\|\nabla\upsilon\|_{\mathrm{L}^2(\Omega)}\left[\left(c_D - C_\varpi\,\lambda(2,d)^2\Lambda(2,d)^\beta\|\nabla\mathbf{v}\|^\beta_{\mathbf{L}^2(\Omega)}\right)\|\nabla\upsilon\|_{\mathrm{L}^2(\Omega)} - C_T\, n\,\sqrt{\mathcal{L}^d(\Omega)}\lambda(2,d)\right].\end{aligned} \tag{4.10}$$

It follows that $\mathcal{P}(\mathbf{v},\upsilon)\cdot(\mathbf{v},\upsilon) > 0$ for $\|\mathbf{v}\|_{\mathbf{V}} = \rho$ and $\|\upsilon\|_V = \varsigma$, and ρ and ς chosen in such a way that

$$\rho > \frac{\Lambda_P(d)}{\nu\, C_K^2}\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)} := K_1 \quad \text{and} \quad \varsigma > \frac{C_T\, n\,\sqrt{\mathcal{L}^d(\Omega)}\lambda(2,d)}{c_D - C_\varpi\,\lambda(2,d)^2\Lambda(2,d)^\beta\rho^\beta} := \frac{K_2}{c_D - K_3\rho^\beta} \quad \text{and} \quad \varsigma > 0.$$

So, ρ must be chosen in such a way that

$$\rho > K_1 \quad \text{and} \quad c_D - K_3\rho^\beta > 0 \Leftrightarrow K_1 < \rho < (c_D K_3)^{\frac{1}{\beta}}\,,$$

which is possible as long as $(c_D K_3)^{\frac{1}{\beta}} - K_1 > 0 \Leftrightarrow c_D > K_1^\beta K_3$, *i.e.* as long as (3.2) is verified with

$$C := C_\varpi\,\lambda(2,d)^2\Lambda(2,d)^\beta C_K^{-2\beta}\Lambda_P(d)^\beta. \tag{4.11}$$

For our purposes, it is enough to consider

$$\rho = \left(K_1^\beta + \rho_0\right)^{\frac{1}{\beta}}, \quad \text{with } 0 < \rho_0 < \frac{c_D - K_1^\beta K_3}{K_3}.$$

The hypotheses of [36, Lemma II.1.4] are thus verified and therefore there exists a solution $(\mathbf{c}_j,\mathbf{d}_j)$, with $\mathbf{c}_j := (c_{1j},\ c_{2j},\ \dots,\ c_{jj})$ and $\mathbf{d}_j := (d_{1j},\ d_{2j},\ \dots,\ d_{jj})$ to the system (4.1)-(4.4).

4.2. **A priori estimates.** Multiplying (4.3) by c_{ij}, adding up the resulting equation between $i = 1$ and $i = j$, and observing that the convective integral term vanishes due to the fact that $\mathbf{div}\,\mathbf{u}_j = 0$, we obtain

$$\int_\Omega (\nu + \nu_T(k_j))\mathbf{D}(\mathbf{u}_j) : \nabla\mathbf{u}_j \, d\mathbf{x} + \int_\Omega \mathbf{f}(\mathbf{u}_j)\cdot\mathbf{u}_j \, d\mathbf{x} = \int_\Omega \mathbf{g}\cdot\mathbf{u}_j \, d\mathbf{x}.$$

Using the symmetry of $\mathbf{D}(\mathbf{u}_j)$ together with the sign condition (2.15) and assumption (2.19), we get

$$\nu \int_\Omega |\mathbf{D}(\mathbf{u}_j)|^2 \, d\mathbf{x} \le \int_\Omega \mathbf{g}\cdot\mathbf{u}_j \, d\mathbf{x}.$$

Then, using a suitable Cauchy's inequality in conjunction with Korn's and Sobolev's inequalities, we establish that

$$\|\nabla\mathbf{u}_j\|^2_{\mathbf{L}^2(\Omega)} \le C, \quad C = C(\nu, d, \Omega, \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}), \tag{4.12}$$

where the positive constant C is independent of j. Due to (4.12) and (3.1), and by means of reflexivity, there exists a subsequence (still denoted by) $\mathbf{u}_j$ and $\mathbf{u}$ in $\mathbf{H}^1_0(\Omega)$ such that

$$\mathbf{u}_j \to \mathbf{u} \quad \text{weakly in } \mathbf{H}^1_0(\Omega), \quad \text{as } j \to \infty. \tag{4.13}$$

By the Sobolev compact imbedding $\mathbf{H}^1_0(\Omega) \hookrightarrow\hookrightarrow \mathbf{L}^\gamma(\Omega)$, with $s \in \left[1, \frac{2d}{d-2}\right)$ if $\gamma \ne 2$ or $\gamma \in [1,\infty)$ if $\gamma = 2$,

$$\mathbf{u}_j \to \mathbf{u} \quad \text{strongly in } \mathbf{L}^\gamma(\Omega), \quad \text{as } j \to \infty. \tag{4.14}$$

Then, due to (4.14) and by Riesz-Fischer theorem, we have, up to a subsequence, that

$$\mathbf{u}_j \to \mathbf{u} \quad \text{a.e. in } \Omega, \quad \text{as } j \to \infty. \tag{4.15}$$

In particular, a refinement of the estimate (4.12) allows us to write

$$\|\nabla\mathbf{u}_j\|_{\mathbf{L}^2(\Omega)} \le \frac{\Lambda_P(d)}{\nu C_K^2}\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}, \tag{4.16}$$

where $\Lambda_P(d)$ is the principal (positive) eigenvalue of the vectorial version of the Laplacian problem (2.22)-(2.23). Then, observing the weak convergence (4.13), we obtain, by passing to the limit inf in (4.16), that

$$\|\nabla\mathbf{u}\|_{\mathbf{L}^2(\Omega)} \le \frac{\Lambda_P(d)}{\nu C_K^2}\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}. \tag{4.17}$$

We will use this estimate later on, for the analysis of the equation for k, when the production term represented by the function P depend on both $\mathbf{u}$ and k.

Then we multiply (4.4) by d_{ij}, we add up the resulting equation between $i = 1$ and $i = j$ and we use the fact that $\mathbf{u}_j \in \mathbf{V}^j$ implies $\mathbf{div}\,\mathbf{u}_j = 0$ and whence the first integral term vanishes. Next we use the sign condition (2.16) and we obtain,

$$\int_\Omega \nu_D(k_j)|\nabla k_j|^2 \, d\mathbf{x} \le \int_\Omega \nu_T(k_j)\,\mathcal{R}_n\Big(|\mathbf{D}(\mathbf{u}_j)|^2\Big)\,k_j \, d\mathbf{x} + \int_\Omega P(\mathbf{u}_j, k_j)k_j \, d\mathbf{x}.$$

4.2.1. ***If*** $P(\mathbf{u}, k) = \pi(\mathbf{u})$ ***a.e. in*** Ω*,* we use the same reasoning we have considered for (4.6) and for (4.7), together with a suitable Cauchy's inequality and we still make use of (4.12), which altogether yield

$$\|\nabla k_j\|^2_{\mathrm{L}^2(\Omega)} \le C, \quad C = C(C_\pi, c_D, C_T, n, \Omega, d, \nu, \beta, \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}), \tag{4.18}$$

for a positive constant C not depending on j.

4.2.2. ***If*** $P(\mathbf{u}, k) = \varpi(\mathbf{u})k$ ***a.e. in*** Ω*,* we argue as we did for (4.18), but using, in this case, (4.9), to obtain

$$c_D\|\nabla k_j\|^2_{\mathrm{L}^2(\Omega)} \le C_T\, n\,\sqrt{\mathcal{L}^d(\Omega)}\lambda(2,d)\|\nabla k_j\|_{\mathrm{L}^2(\Omega)} + C_\varpi\,\lambda(2,d)^2\Lambda(2,d)^\beta\left(\frac{\Lambda_P(d)}{\nu C_K^2}\right)^\beta \|\mathbf{g}\|^\beta_{\mathbf{L}^2(\Omega)}\|\nabla k_j\|^2_{\mathrm{L}^2(\Omega)}.$$

This yields

$$\|\nabla k_j\|_{\mathrm{L}^2(\Omega)} \le C_T\, n\,\sqrt{\mathcal{L}^d(\Omega)}\lambda(2,d)\left(c_D - C_\varpi\,\lambda(2,d)^2\Lambda(2,d)^\beta\left(\frac{\Lambda_P(d)}{\nu C_K^2}\right)^\beta \|\mathbf{g}\|^\beta_{\mathbf{L}^2(\Omega)}\right)^{-1}, \tag{4.19}$$

and, by using assumption (3.2), with C defined at (4.11), we can readily see that the right-hand side of (4.19) is a positive constant independent of j.

By means of reflexivity and due to the Sobolev compact imbedding $\mathrm{H}^1_0(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}^\gamma(\Omega)$, with $\gamma \in \left[1, \frac{2d}{d-2}\right)$ if $\gamma \neq 2$ or $\gamma \in [1, \infty)$ if $\gamma = 2$, we have, in view of (4.18), or of (4.19), and up to a subsequence, that

$$k_j \to k \quad \text{weakly in } \mathrm{H}^1_0(\Omega), \quad \text{as } j \to \infty, \tag{4.20}$$

$$k_j \to k \quad \text{strongly in } \mathrm{L}^\gamma(\Omega), \quad \text{as } j \to \infty. \tag{4.21}$$

Due to (4.21), we have by Riesz-Fischer theorem, and up to a subsequence, that

$$k_j \to k \quad \text{a.e. in } \Omega, \quad \text{as } j \to \infty. \tag{4.22}$$

4.3. **Passing to the limit $j \to \infty$.** We start by passing to the limit $j \to \infty$ the integral equality (4.3). The convergence of the last term of (4.3) follows from (4.13) and assumption (3.1). In what follows, we just consider the case $d \geq 3$ (the case $d = 2$ is simpler).

Now let us fix our attention on the convergence of the third term of (4.3). Since $\mathbf{f}$ is continuous on $\mathbf{u}$ (see (2.1)), we have by virtue of (4.15) that

$$\mathbf{f}(\mathbf{u}_j) \to \mathbf{f}(\mathbf{u}) \quad \text{a.e. in } \Omega, \quad \text{as } j \to \infty. \tag{4.23}$$

On the other hand, using (2.8) and Sobolev's inequality together with (4.12), it can be proved that

$$\|\mathbf{f}(\mathbf{u}_j)\|_{\mathbf{L}^{\frac{2d}{d+2}}(\Omega)} \leq C, \tag{4.24}$$

for some positive constant C independent of j. Owing to (4.23) and (4.24), we obtain

$$\mathbf{f}(\mathbf{u}_j) \to \mathbf{f}(\mathbf{u}) \text{ weakly in } \mathbf{L}^{\frac{2d}{d+2}}(\Omega), \quad \text{as } j \to \infty, \tag{4.25}$$

By the Sobolev imbedding, $\mathbf{v}_i \in \mathbf{H}^1_0(\Omega) \hookrightarrow \mathbf{L}^{\frac{2d}{d-2}}(\Omega)$ and since $\left(\frac{2d}{d+2}\right)^{-1} + \left(\frac{2d}{d-2}\right)^{-1} = 1$, from (4.25), we have

$$\int_\Omega \mathbf{f}(\mathbf{u}_j) \cdot \mathbf{v}_i \to \int_\Omega \mathbf{f}(\mathbf{u}) \cdot \mathbf{v}_i, \quad \text{as } j \to \infty, \quad \text{for all } i \geq 1. \tag{4.26}$$

For the first term of (4.3), we start by observing that, due to (1.12) and (1.15), we can write

$$\int_\Omega ((\mathbf{u}_j \cdot \boldsymbol{\nabla})\mathbf{u}_j) \cdot \mathbf{v}_i \, d\mathbf{x} = -\int_\Omega \mathbf{u}_j \otimes \mathbf{u}_j : \boldsymbol{\nabla}\mathbf{v}_i \, d\mathbf{x}.$$

From (4.12), this used together with the Sobolev imbedding $\mathbf{H}^1_0(\Omega) \hookrightarrow \mathbf{L}^{\frac{2d}{d-2}}(\Omega)$, and (4.15), we have

$$\|\mathbf{u}_j \otimes \mathbf{u}_j\|_{\mathbf{L}^{\frac{d}{d-2}}(\Omega)} \leq C \quad \text{and} \quad \mathbf{u}_j \otimes \mathbf{u}_j \to \mathbf{u} \otimes \mathbf{u} \text{ a.e. in } \Omega, \text{ as } j \to \infty, \tag{4.27}$$

where C is a positive constant not depending on j. Consequently, (4.27) yields

$$\mathbf{u}_j \otimes \mathbf{u}_j \to \mathbf{u} \otimes \mathbf{u} \text{ weakly in } \mathbf{L}^{\frac{d}{d-2}}(\Omega), \text{ as } j \to \infty. \tag{4.28}$$

Then, since $\boldsymbol{\nabla}\mathbf{v}_i \in \mathbf{L}^2(\Omega) \subset \mathbf{L}^{\frac{d}{2}}(\Omega)$ for $d \leq 4$ (and bounded Ω), we have, by virtue of (4.28) and once that $\left(\frac{d}{d-2}\right)^{-1} + \left(\frac{d}{2}\right)^{-1} = 1$,

$$\int_\Omega \mathbf{u}_j \otimes \mathbf{u}_j : \boldsymbol{\nabla}\mathbf{v}_i \, d\mathbf{x} \to \int_\Omega \mathbf{u} \otimes \mathbf{u} : \boldsymbol{\nabla}\mathbf{v}_i \, d\mathbf{x}, \quad \text{as } j \to \infty, \quad \text{for all } i \geq 1. \tag{4.29}$$

Let us now show the convergence of the second term of (4.3). We first observe that (2.2), (2.19) and (4.22) imply

$$(\nu + \nu_T(k_j))\boldsymbol{\nabla}\mathbf{v}_i \to (\nu + \nu_T(k))\boldsymbol{\nabla}\mathbf{v}_i \quad \text{a.e. in } \Omega, \quad \text{as } j \to \infty,$$
$$\left|(\nu + \nu_T(k_j))\boldsymbol{\nabla}\mathbf{v}_i\right| \leq (\nu + C_T)|\boldsymbol{\nabla}\mathbf{v}_i|.$$

Then, since $\boldsymbol{\nabla}\mathbf{v}_i \in \mathbf{L}^2(\Omega)$, we have, by Lebesgue's dominated convergence theorem,

$$(\nu + \nu_T(k_j))\boldsymbol{\nabla}\mathbf{v}_i \to (\nu + \nu_T(k))\boldsymbol{\nabla}\mathbf{v}_i \quad \text{strongly in } \mathbf{L}^2(\Omega), \quad \text{as } j \to \infty. \tag{4.30}$$

Then, from (4.13) and (4.30), we can prove that

$$\int_\Omega (\nu + \nu_T(k_j))\mathbf{D}(\mathbf{u}_j) : \boldsymbol{\nabla}\mathbf{v}_i \, d\mathbf{x} \to \int_\Omega (\nu + \nu_T(k))\mathbf{D}(\mathbf{u}) : \boldsymbol{\nabla}\mathbf{v}_i \, d\mathbf{x}, \quad \text{as } j \to \infty, \quad \text{for all } i \geq 1. \tag{4.31}$$

The convergences (4.26), (4.29) and (4.31) imply that we can pass to the limit $j \to \infty$ in the approximate system (4.3) and thus we obtain

$$\begin{aligned}\int_\Omega ((\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u})\cdot\mathbf{v}_i\,d\mathbf{x} + \int_\Omega (\nu+\nu_T(k))\mathbf{D}(\mathbf{u}):\boldsymbol{\nabla}\mathbf{v}_i\,d\mathbf{x} + \int_\Omega \mathbf{f}(\mathbf{u})\cdot\mathbf{v}_i\,d\mathbf{x}\\ = \int_\Omega \mathbf{g}\cdot\mathbf{v}_i\,d\mathbf{x}\end{aligned} \tag{4.32}$$

for all $i \geq 1$. Using the linearity of (4.32) in $\mathbf{v}_i$ and the density of the finite linear combinations of the system $\{\mathbf{v}_i\}_{i=1}^{\infty}$ in $\mathbf{V}$, we deduce that (4.32) holds true in the whole space $\mathbf{V}$, that is

$$\begin{aligned}\int_\Omega ((\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u})\cdot\mathbf{v}\,d\mathbf{x} + \int_\Omega (\nu+\nu_T(k))\mathbf{D}(\mathbf{u}):\boldsymbol{\nabla}\mathbf{v}\,d\mathbf{x} + \int_\Omega \mathbf{f}(\mathbf{u})\cdot\mathbf{v}\,d\mathbf{x}\\ = \int_\Omega \mathbf{g}\cdot\mathbf{v}\,d\mathbf{x}\end{aligned} \tag{4.33}$$

for all $\mathbf{v}\in\mathbf{V}$. This allows us to take $\mathbf{v}=\mathbf{u}$ as a test function in (4.33), which yields

$$\int_\Omega (\nu+\nu_T(k))|\mathbf{D}(\mathbf{u})|^2\,d\mathbf{x} + \int_\Omega \mathbf{f}(\mathbf{u})\cdot\mathbf{u}\,d\mathbf{x} = \int_\Omega \mathbf{g}\cdot\mathbf{u}\,d\mathbf{x}. \tag{4.34}$$

Taking $\mathbf{v}_i = \mathbf{u}_j$ in (4.3), we also have the equality

$$\int_\Omega (\nu+\nu_T(k_j))|\mathbf{D}(\mathbf{u}_j)|^2\,d\mathbf{x} + \int_\Omega \mathbf{f}(\mathbf{u}_j)\cdot\mathbf{u}_j\,d\mathbf{x} = \int_\Omega \mathbf{g}\cdot\mathbf{u}_j\,d\mathbf{x}. \tag{4.35}$$

In (4.34)-(4.35), we have used the facts that $\mathbf{u}$ and $\mathbf{u}_j$ are solenoidal and $\mathbf{D}(\mathbf{u})$ and $\mathbf{D}(\mathbf{u}_j)$ are symmetric. Then, using (4.34) and (4.35) together with (4.14), (4.21) and (4.26), we have

$$\lim_{j\to\infty}\int_\Omega (\nu+\nu_T(k_j))|\mathbf{D}(\mathbf{u}_j)|^2\,d\mathbf{x} = \int_\Omega \mathbf{g}\cdot\mathbf{u}\,d\mathbf{x} - \int_\Omega \mathbf{f}(\mathbf{u})\cdot\mathbf{u}\,d\mathbf{x} = \int_\Omega (\nu+\nu_T(k))|\mathbf{D}(\mathbf{u})|^2\,d\mathbf{x}. \tag{4.36}$$

On the other hand, arguing as we did for (4.31), we can prove that

$$(\nu+\nu_T(k_j))^{\frac{1}{2}}\mathbf{D}(\mathbf{u}_j) \to (\nu+\nu_T(k))^{\frac{1}{2}}\mathbf{D}(\mathbf{u}) \quad \text{weakly in } \mathbf{L}^2(\Omega), \quad \text{as } j\to\infty. \tag{4.37}$$

Combining (4.36) and (4.37), it yields

$$(\nu+\nu_T(k_j))^{\frac{1}{2}}\mathbf{D}(\mathbf{u}_j) \to (\nu+\nu_T(k))^{\frac{1}{2}}\mathbf{D}(\mathbf{u}) \quad \text{strongly in } \mathbf{L}^2(\Omega), \quad \text{as } j\to\infty. \tag{4.38}$$

Now, we observe that, in view of (2.19), we have

$$\begin{aligned}&\int_\Omega |\mathbf{D}(\mathbf{u}_j)-\mathbf{D}(\mathbf{u})|^2\,d\mathbf{x} \leq\\ &\frac{1}{\nu}\left[\int_\Omega (\nu+\nu_T(k_j))|\mathbf{D}(\mathbf{u}_j)|^2\,d\mathbf{x} - 2\int_\Omega (\nu+\nu_T(k_j))\mathbf{D}(\mathbf{u}_j):\mathbf{D}(\mathbf{u})\,d\mathbf{x} + \int_\Omega (\nu+\nu_T(k_j))|\mathbf{D}(\mathbf{u})|^2\,d\mathbf{x}\right].\end{aligned}$$

Then, using (4.38) in the first term, (4.37) in the second and reasoning, in the third term, as we did at (4.30), we can prove that

$$\mathbf{D}(\mathbf{u}_j) \to \mathbf{D}(\mathbf{u}) \quad \text{strongly in } \mathbf{L}^2(\Omega), \quad \text{as } j\to\infty. \tag{4.39}$$

Finally, by Riesz-Fisher's theorem, we have, up to a subsequence,

$$\mathbf{D}(\mathbf{u}_j) \to \mathbf{D}(\mathbf{u}) \quad \text{a.e. in } \Omega, \quad \text{as } j\to\infty. \tag{4.40}$$

We will now pass to the limit $j\to\infty$ the integral equality (4.4). Here again and in what follows, we just consider the case $d \geq 3$ (the case $d = 2$ is simpler). To pass the first term of this equality to the limit, we can argue as we did for the convective term of the Navier-Stokes equations (see (4.29)). This yields

$$\int_\Omega (\mathbf{u}_j\cdot\boldsymbol{\nabla}k_j)\upsilon_i\,d\mathbf{x} \to \int_\Omega (\mathbf{u}\cdot\boldsymbol{\nabla}k)\upsilon_i\,d\mathbf{x}, \quad \text{as } j\to\infty, \quad \text{for all } i\geq 1. \tag{4.41}$$

We now study the convergence of the second term in (4.4). Arguing as we did for (4.30), but using here (2.20) instead of (2.19), we also have

$$\nu_D(k_j)\boldsymbol{\nabla}\mathbf{v}_i \to \nu_D(k)\boldsymbol{\nabla}\mathbf{v}_i \quad \text{strongly in } \mathbf{L}^2(\Omega). \tag{4.42}$$

Then, from (4.20) and (4.42), we also can prove that

$$\int_\Omega \nu_D(k_j)\nabla k_j\cdot\nabla\upsilon_i\,d\mathbf{x}\to\int_\Omega \nu_D(k)\nabla k\cdot\nabla\upsilon_i\,d\mathbf{x},\quad \text{as } j\to\infty,\quad \text{for all } i\geq 1. \tag{4.43}$$

Next, we justify the convergence of the third term. Due to assumption (2.2) and to (4.22), we have

$$\varepsilon(k_j)\to\varepsilon(k)\quad \text{a.e. in }\Omega,\quad \text{as } j\to\infty. \tag{4.44}$$

On the other, using assumption (2.9) together with Sobolev's inequality and (4.18), or (4.19), it can be proved that

$$\|\varepsilon(k_j)\|_{\mathrm{L}^{\frac{2d}{d+2}}(\Omega)}\leq C, \tag{4.45}$$

for some positive constant C not depending on j. Owing to (4.44) and (4.45), we have

$$\varepsilon(k_j)\to\varepsilon(k)\quad \text{weakly in } \mathrm{L}^{\frac{2d}{d+2}}(\Omega),\quad \text{as } j\to\infty. \tag{4.46}$$

Then, since $\upsilon_i\in \mathrm{H}^1_0(\Omega)\hookrightarrow \mathrm{L}^{\frac{2d}{d-2}}(\Omega)$ and once that $\left(\frac{2d}{d+2}\right)^{-1}+\left(\frac{2d}{d-2}\right)^{-1}=1$, we have, in view of (4.46), that

$$\int_\Omega \varepsilon(k_j)\upsilon_i\,d\mathbf{x}\to\int_\Omega \varepsilon(k)\upsilon_i\,d\mathbf{x},\quad \text{as } j\to\infty,\quad \text{for all } i\geq 1. \tag{4.47}$$

For the fourth term, we just consider the continuity of ν_T (see (2.2)) and the definition of the regularization $\mathcal{R}_n$ (see (3.7)) together with (4.22) and (4.40) to obtain that

$$\int_\Omega \nu_T(k_j)\,\mathcal{R}_n\Big(|\mathbf{D}(\mathbf{u}_j)|^2\Big)\,\upsilon_i\,d\mathbf{x}\to\int_\Omega \nu_T(k)\,\mathcal{R}_n\Big(|\mathbf{D}(\mathbf{u})|^2\Big)\,\upsilon_i\,d\mathbf{x},\quad \text{as } j\to\infty,\quad \text{for all } i\geq 1. \tag{4.48}$$

If $P(\mathbf{u},k)=\pi(\mathbf{u})$ a.e. in Ω or $P(\mathbf{u},k)=\varpi(\mathbf{u})k$ a.e. in Ω, it remains to pass to the limit $j\to\infty$ the correspondingly integral terms.

Assumption (2.11) together with (4.15) and (4.22) imply that

$$\pi(\mathbf{u}_j)\to\pi(\mathbf{u})\quad \text{a.e. in }\Omega,\quad \text{as } j\to\infty, \tag{4.49}$$

$$\varpi(\mathbf{u}_j)k_j\to\varpi(\mathbf{u})k\quad \text{a.e. in }\Omega,\quad \text{as } j\to\infty. \tag{4.50}$$

By using assumption (2.13), or (2.14), Hölder's inequality (just in the second case) and Sobolev's inequality together with (4.12) and (4.18), or (4.19), it can be proved, for $d\geq 3$ (for $d=2$ is easier), that

$$\|\pi(\mathbf{u}_j)\|_{\mathrm{L}^{\frac{2d}{d+2}}(\Omega)}\leq C_\pi\|\mathbf{u}_j\|^\beta_{\mathbf{L}^{\frac{2d}{d+2}}(\Omega)}\leq C_1, \tag{4.51}$$

$$\|\varpi(\mathbf{u}_j)k_j\|_{\mathrm{L}^{\frac{2d}{d+2}}(\Omega)}\leq C_\varpi\|\mathbf{u}_j\|^\beta_{\mathbf{L}^{\frac{2d}{d-2}}(\Omega)}\|k_j\|_{\mathrm{L}^{\frac{2d}{d-2}}(\Omega)}\leq C_2, \tag{4.52}$$

for some positive constants C_1 and C_2 not depending on j. Now, (4.49) and (4.51), or (4.50) and (4.52), yield

$$\pi(\mathbf{u}_j)\to\pi(\mathbf{u})\quad \text{weakly in } \mathrm{L}^{\frac{2d}{d+2}}(\Omega),\quad \text{as } j\to\infty, \tag{4.53}$$

$$\varpi(\mathbf{u}_j)k_j\to\varpi(\mathbf{u})k\quad \text{weakly in } \mathrm{L}^{\frac{2d}{d+2}}(\Omega),\quad \text{as } j\to\infty. \tag{4.54}$$

Then, since again $\upsilon_i\in \mathrm{H}^1_0(\Omega)\hookrightarrow \mathrm{L}^{\frac{2d}{d-2}}(\Omega)$, (4.53) and (4.54) yield

$$\int_\Omega \pi(\mathbf{u}_j)\upsilon_i\,d\mathbf{x}\to\int_\Omega \pi(\mathbf{u})\upsilon_i\,d\mathbf{x},\quad \text{as } j\to\infty,\quad \text{for all } i\geq 1, \tag{4.55}$$

$$\int_\Omega \varpi(\mathbf{u}_j)k_j\upsilon_i\,d\mathbf{x}\to\int_\Omega \varpi(\mathbf{u})k\upsilon_i\,d\mathbf{x},\quad \text{as } j\to\infty,\quad \text{for all } i\geq 1. \tag{4.56}$$

The convergences (4.41), (4.43), (4.47), (4.48) and (4.55), or (4.56), assure that we can pass to the limit $j\to\infty$ in the approximate system (4.4) to obtain

$$\begin{aligned}&\int_\Omega(\mathbf{u}\cdot\nabla k)\upsilon_i\,d\mathbf{x}+\int_\Omega \nu_D(k)\nabla k\cdot\nabla\upsilon_i\,d\mathbf{x}+\int_\Omega\varepsilon(k)\upsilon_i\,d\mathbf{x}\\ =&\int_\Omega \nu_T(k)\,\mathcal{R}_n\Big(|\mathbf{D}(\mathbf{u})|^2\Big)\,\upsilon_i\,d\mathbf{x}+\int_\Omega P(\mathbf{u},k)\,\upsilon_i\,d\mathbf{x}\end{aligned} \tag{4.57}$$

for all $i\geq 1$.

We have thus proved that, for each $n \in \mathbb{N}$, there exists a weak solution $(\mathbf{u}_n, k_n) \in \mathbf{V} \times \mathrm{H}_0^1(\Omega)$ to the problem (3.3)-(3.6) and such that

$$\begin{aligned} &\int_\Omega (\mathbf{u}_n \cdot \boldsymbol{\nabla})\mathbf{u}_n \cdot \mathbf{v}\, d\mathbf{x} + \int_\Omega (\nu + \nu_T(k_n))\mathbf{D}(\mathbf{u}_n) : \boldsymbol{\nabla}\mathbf{v}\, d\mathbf{x} + \int_\Omega \mathbf{f}(\mathbf{u}_n)\cdot \mathbf{v}\, d\mathbf{x} \\ &= \int_\Omega \mathbf{g}\cdot\mathbf{v}\, d\mathbf{x} \end{aligned} \tag{4.58}$$

and

$$\begin{aligned} &\int_\Omega (\mathbf{u}_n \cdot \boldsymbol{\nabla} k_n) v\, d\mathbf{x} + \int_\Omega \nu_D(k_n)\boldsymbol{\nabla} k_n \cdot \boldsymbol{\nabla} v\, d\mathbf{x} + \int_\Omega \varepsilon(k_n) v\, d\mathbf{x} \\ &= \int_\Omega \nu_T(k_n)\, \mathcal{R}_n\Big(|\mathbf{D}(\mathbf{u}_n)|^2\Big)\, v\, d\mathbf{x} + \int_\Omega P(\mathbf{u}_n, k_n)\, v\, d\mathbf{x} \end{aligned} \tag{4.59}$$

hold for all $(\mathbf{v}, v) \in \mathbf{V}^j \times V^j$ and all $j \geq 1$. By linearity and density these relations hold for all $(\mathbf{v}, v) \in \mathbf{V} \times \mathrm{V}$, and by continuity they hold for all $(\mathbf{v}, v) \in \mathbf{V}\times\mathrm{H}_0^1(\Omega)$ due to the ranges of α, θ and β set forth at (2.8), (2.9) and at (2.13)-(2.14).

4.4. **To show that $k \geq 0$ a.e. in Ω.** Let us decompose k_n as $k_n = k_n^+ - k_n^-$, where $k_n^+ := \max\{0, k_n\}$ and $k_n^- := -\min\{0, k_n\}$. Since $k_n \in \mathrm{H}_0^1(\Omega)$ implies that $k_n^- \in \mathrm{H}_0^1(\Omega)$, we can take $v = -k_n^-$ in (4.59), where we consider (2.17) for the expression of the turbulent dissipation function, to get

$$\begin{aligned} &-\int_\Omega (\mathbf{u}_n \cdot \boldsymbol{\nabla} k_n) k_n^-\, d\mathbf{x} - \int_\Omega \nu_D(k_n)\boldsymbol{\nabla} k_n \cdot \boldsymbol{\nabla} k_n^-\, d\mathbf{x} - \int_\Omega e(k_n) k_n k_n^-\, d\mathbf{x} \\ &= -\int_\Omega \nu_T(k_n)\, \mathcal{R}_n\Big(|\mathbf{D}(\mathbf{u}_n)|^2\Big)\, k_n^-\, d\mathbf{x} - \int_\Omega P(\mathbf{u}_n, k_n) k_n^-\, d\mathbf{x}. \end{aligned}$$

Observing the properties of k_n^+ and k_n^- (see *e.g.* [9, p. 239]) and that $\mathcal{R}_n\Big(|\mathbf{D}(\mathbf{u}_n)|^2\Big) \geq 0$ (see (3.7)), we can use the fact that $\mathbf{u}_n \in \mathbf{V}$ implies $\mathbf{div}\, \mathbf{u}_n = 0$ together with (2.18) and with the assumptions (2.19)-(2.20), to prove that

$$c_D \int_\Omega |\boldsymbol{\nabla} k_n^-|^2\, d\mathbf{x} \leq -\int_\Omega P(\mathbf{u}_n, k_n) k_n^-\, d\mathbf{x}. \tag{4.60}$$

4.4.1. ***If* $P(\mathbf{u}_n, k_n) = \pi(\mathbf{u}_n)$ *a.e. in* Ω,** the right-hand side of (4.60) is bounded by zero since, by the assumption (2.12), $\pi(\mathbf{u}_n) \geq 0$. Then, by the Sobolev imbedding and due to the positiveness of c_D (see (2.20)), we have $\|k^-\|^2_{\mathrm{L}^2(\Omega)} \leq \liminf_{n\to\infty} \|k_n^-\|^2_{\mathrm{L}^2(\Omega)} \leq 0$. This proves that $k^- = 0$ a.e. in Ω and, consequently, $k \geq 0$ a.e. in Ω.

4.4.2. ***If* $P(\mathbf{u}_n, k_n) = \varpi(\mathbf{u}_n)k_n$ *a.e. in* Ω,** we decompose k_n appearing here as $k_n = k_n^+ - k_n^-$, and using the fact that $k_n^+ k_n^- = 0$ together with (2.14), Hölder's inequality and both scalar and vectorial versions of Sobolev's inequality (2.26), we obtain from (4.60)

$$c_D \int_\Omega |\boldsymbol{\nabla} k_n^-|^2\, d\mathbf{x} \leq C_\varpi\, \lambda(2,d)^2 \Lambda(2,d)^\beta \|\boldsymbol{\nabla}\mathbf{u}_n\|^\beta_{\mathbf{L}^2(\Omega)} \|\boldsymbol{\nabla} k_n^-\|^2_{\mathrm{L}^2(\Omega)}$$

Then, using (4.17), we obtain

$$\left(c_D - C\left(\frac{\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}}{\nu}\right)^\beta\right) \int_\Omega |\boldsymbol{\nabla} k_n^-|^2\, d\mathbf{x} \leq 0,$$

where C is the constant defined at (4.11). Arguing as in the previous case, we have $\|k^-\|^2_{\mathrm{L}^2(\Omega)} \leq \liminf_{n\to\infty} \|k_n^-\|^2_{\mathrm{L}^2(\Omega)} \leq 0$, as long as the assumption (3.2) holds, and we also end up with $\|k^-\|^2_{\mathrm{L}^2(\Omega)} \leq 0$.

Finally, $\varepsilon(k) \geq 0$ a.e. in Ω is a direct consequence of applying the assumption (2.16) together with the above conclusion. The proof of Proposition 3.1 is now concluded. □

In the next section we follow with the proof of Theorem 3.1

5. End of the proof of Theorem 3.1

From Proposition 3.1, we know that, for each $n \in \mathbb{N}$, there exists a weak solution $(\mathbf{u}_n, k_n) \in \mathbf{V} \times \mathrm{H}^1_0(\Omega)$ to the problem (3.3)-(3.6) and such that (4.58)-(4.59) hold. The end of the proof of Theorem 3.1 will be split into two parts for the sake of comprehension.

5.1. **A priori estimates.** We start by obtaining an estimate for $\mathbf{u}_n$. Since the sought solutions and the test functions are in the same function space, we can take $\mathbf{v} = \mathbf{u}_n$ in (4.58) and we obtain, after we use the symmetry of $\mathbf{D}(\mathbf{u}_n)$ and the fact that $\mathbf{u}_n \in \mathbf{V}$ implies that $\mathbf{div}\,\mathbf{u}_n = 0$,

$$\int_\Omega (\nu + \nu_T(k_n))|\mathbf{D}(\mathbf{u}_n)|^2\, d\mathbf{x} + \int_\Omega \mathbf{f}(\mathbf{u}_n)\cdot \mathbf{u}_n\, d\mathbf{x} = \int_\Omega \mathbf{g}\cdot\mathbf{u}_n\, d\mathbf{x}. \tag{5.1}$$

Proceeding as we did for (4.16), we obtain

$$\|\boldsymbol{\nabla}\mathbf{u}_n\|_{\mathbf{L}^2(\Omega)} \le \frac{\Lambda_P(d)}{\nu C_K^2}\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}. \tag{5.2}$$

Then, arguing as we did for (4.13)-(4.15) and in view of (5.2) and of the assumption (3.1), we have

$$\mathbf{u}_n \to \mathbf{u} \quad \text{weakly in } \mathbf{H}^1_0(\Omega), \quad \text{as } n\to\infty, \tag{5.3}$$

$$\mathbf{u}_n \to \mathbf{u} \quad \text{strongly in } \mathbf{L}^\gamma(\Omega), \quad \text{as } n\to\infty, \quad \text{for } \gamma\in\left[1,\tfrac{2d}{d-2}\right) \text{ if } d\neq 2, \text{ or any } \gamma\in[1,\infty) \text{ if } d=2 \tag{5.4}$$

$$\mathbf{u}_n \to \mathbf{u} \quad \text{a.e. in } \Omega, \quad \text{as } n\to\infty. \tag{5.5}$$

The achievement of an *a priori* estimate for k_n, independent of n, is more subtle. Note that the reasoning used to get (4.18) or (4.19) cannot be done here, because the constant there depends on n. Due to this extra difficulty, we consider the following special test function in the spirit of [7, 29, 33]

$$\varphi(k_n) := 1 - \frac{1}{(1+k_n)^\delta}, \tag{5.6}$$

where δ is a positive constant to be defined later on and such that $\varphi \in \mathrm{W}^{1,q'}(\Omega) \hookrightarrow C^{0,\delta}(\Omega)$, *i.e.* with $q' > d$. Observe that $\varphi(k_n)$ satisfies to

$$0 \le \varphi(k_n) \le 1, \quad \boldsymbol{\nabla}\varphi(k_n) = \delta\frac{\boldsymbol{\nabla}k_n}{(1+k_n)^{\delta+1}} \tag{5.7}$$

and therefore $\varphi(k_n) \in \mathrm{H}^1_0(\Omega)$. Thus we may take $v = \varphi(k_n)$ in (4.59) to get

$$\begin{aligned}&\int_\Omega (\mathbf{u}_n\cdot\boldsymbol{\nabla}k_n)\varphi(k_n)\, d\mathbf{x} + \int_\Omega \nu_D(k_n)\boldsymbol{\nabla}k_n\cdot\boldsymbol{\nabla}\varphi(k_n)\, d\mathbf{x} + \int_\Omega \varepsilon(k_n)\varphi(k_n)\, d\mathbf{x}\\ &= \int_\Omega \nu_T(k_n)\,\mathcal{R}_n\Big(|\mathbf{D}(\mathbf{u}_n)|^2\Big)\,\varphi(k_n)\, d\mathbf{x} + \int_\Omega P(\mathbf{u}_n,k_n)\,\varphi(k_n)\, d\mathbf{x},\end{aligned} \tag{5.8}$$

where the first term vanishes due to the fact that $\mathbf{u}_n \in \mathbf{V}$ implies that $\mathbf{div}\,\mathbf{u}_n = 0$,

$$\int_\Omega (\mathbf{u}_n\cdot\boldsymbol{\nabla}k_n)\varphi(k_n)\, d\mathbf{x} = \int_\Omega \mathbf{u}_n\cdot\boldsymbol{\nabla}\Phi(k_n)\, d\mathbf{x} = 0, \quad \Phi(y) := \int_0^y \varphi(\tau)\, d\tau. \tag{5.9}$$

We observe that the third term of (5.8) is nonnegative once that $\varphi(k_n) \ge 0$ and due to the fact (proved in Proposition 3.1) that $\varepsilon(k_n) \ge 0$ a.e. in Ω. As a consequence of this and of (5.9), and attending to (3.8) and (5.7) and observing that $\varphi(k_n) \le 1$, we obtain

$$\delta\int_\Omega \nu_D(k_n)\frac{|\boldsymbol{\nabla}k_n|^2}{(1+k_n)^{1+\delta}}\, d\mathbf{x} \le \int_\Omega \nu_T(k_n)\,|\mathbf{D}(\mathbf{u}_n)|^2\, d\mathbf{x} + \int_\Omega |P(\mathbf{u}_n,k_n)|\, d\mathbf{x}.$$

5.1.1. ***If*** $P(\mathbf{u},k)=\pi(\mathbf{u})$ ***a.e. in*** Ω, by the assumptions (2.19)-(2.20) and (2.13), due to the Hölder inequality and to the vectorial version of the Sobolev inequality (2.26), we have

$$\delta\int_\Omega \frac{|\nabla k_n|^2}{(1+k_n)^{1+\delta}}\,d\mathbf{x}\le C_1\|\nabla\mathbf{u}_n\|^2_{\mathbf{L}^2(\Omega)}+C_2\|\nabla\mathbf{u}_n\|^\beta_{\mathbf{L}^2(\Omega)},\tag{5.10}$$

where C_1 and C_2 are positive constants not depending on n. Now, arguing as in [29], we use (5.10) together with Hölder and Sobolev inequalities as follows

$$\begin{aligned}\int_\Omega|\nabla k_n|^q\,d\mathbf{x}\le&\left(\int_\Omega\frac{|\nabla k_n|^2}{(1+k_n)^{1+\delta}}\,d\mathbf{x}\right)^{\frac{q}{2}}\left(\int_\Omega(1+k_n)^{\frac{(1+\delta)q}{2-q}}\,d\mathbf{x}\right)^{\frac{2-q}{2}}\\ \le&\frac{C_1}{\delta}\|\nabla\mathbf{u}_n\|^2_{\mathbf{L}^2(\Omega)}+\frac{C_2}{\delta}\|\nabla\mathbf{u}_n\|^\beta_{\mathbf{L}^2(\Omega)}+C_3\|\nabla k_n\|^{\frac{(1+\delta)q}{2}}_{\mathrm{L}^q(\Omega)}+C_4,\end{aligned}\tag{5.11}$$

where C_3 and C_4 are also positive constants not depending on n. The last application of Hölder and Sobolev inequalities are possible under the assumptions that

$$\frac{2}{q}\ge1\Leftrightarrow q\le2\quad\text{and}\quad\left(\delta>0\text{ and }\frac{(1+\delta)q}{2-q}\le q*\Leftrightarrow\delta\le\frac{(2-q)d}{d-q}-1\right)\Rightarrow q<d'.$$

Since $d\ge2$, we have, in view of this, that it must be $q<d'$, condition that holds by Definition 2.1. Now, using suitable Young's inequalities in (5.11) together with (5.2) and assumption (3.1), we obtain

$$\int_\Omega|\nabla k_n|^q\,d\mathbf{x}\le C,\quad C=C(\nu,\beta,c_D,C_T,C_\pi,d,q,\Omega,\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}),\tag{5.12}$$

where C is a positive constant not depending on n. Observe that this reasoning is possible for a choice of δ satisfying to $0<\delta<\min\left\{\frac{(2-q)d}{d-q}-1,1\right\}=\frac{(2-q)d}{d-q}-1$ for $d\ge2$. Then, arguing as we did for (4.20)-(4.22) and in view of (5.12) and of the assumption (3.1), we have

$$k_n\to k\quad\text{weakly in }\mathrm{W}^{1,q}_0(\Omega),\quad\text{as }n\to\infty,\quad\text{for }q<d',\tag{5.13}$$

$$k_n\to k\quad\text{strongly in }\mathrm{L}^\gamma(\Omega),\quad\text{as }n\to\infty,\quad\text{for all }\gamma\text{: }1\le\gamma<q^*,\tag{5.14}$$

$$k_n\to k\quad\text{a.e. in }\Omega,\quad\text{as }n\to\infty.\tag{5.15}$$

5.1.2. ***If*** $P(\mathbf{u},k)=\varpi(\mathbf{u})k$ ***a.e. in*** Ω, we argue as we did for (5.10), but using here (2.14) instead of (2.13) and still the Sobolev inequality (2.24), to obtain

$$\delta\int_\Omega\frac{|\nabla k_n|^2}{(1+k_n)^{1+\delta}}\,d\mathbf{x}\le C_1\|\nabla\mathbf{u}_n\|^2_{\mathbf{L}^2(\Omega)}+C_2\|\nabla\mathbf{u}_n\|^\beta_{\mathbf{L}^2(\Omega)}\|\nabla k_n\|_{\mathrm{L}^q(\Omega)},\tag{5.16}$$

where C_1 and C_2 are positive constants independent of n. Then by a similar reasoning to that one we have used for (5.12), and using, in addition, a suitable Young's inequality, we can show that

$$\int_\Omega|\nabla k_n|^q\,d\mathbf{x}\le C,\quad C=C(\nu,\beta,c_D,C_T,C_\varpi,d,q,\Omega,\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}),\tag{5.17}$$

where C is also a positive constant not depending on n. Due to the assumption (3.1), the convergence results (5.13)-(5.15) are also valid here as a consequence of (5.17).

Now we can pass to the limit $n\to\infty$ almost all integral terms of (4.58)-(4.59) by arguing analogously as we did in the previous section. The only term that requires a special treatment is the one involving $\mathcal{R}_n$, because we do not know whether this term remains bounded as $n\to\infty$.

5.2. **Passing $\mathcal{R}_n\left(|\mathbf{D}(\mathbf{u}_n)|^2\right)$ to the limit $n\to\infty$.** Let us thus look to the fourth term of (5.8). Using the definition of $\mathcal{R}_n$ (see (3.7)-(3.8)), we have

$$\begin{aligned}&\int_\Omega\left|\left(\nu_T(k_n)\mathcal{R}_n\left(|\mathbf{D}(\mathbf{u}_n)|^2\right)-\nu_T(k)|\mathbf{D}(\mathbf{u})|^2\right)v\right|\,d\mathbf{x}\\ \le&\int_\Omega\left|\nu_T(k_n)|\mathbf{D}(\mathbf{u}_n)|^2-\nu_T(k)|\mathbf{D}(\mathbf{u})|^2\right|\,|v|\,d\mathbf{x}+\int_\Omega\frac{1}{n}\frac{\nu_T(k)|\mathbf{D}(\mathbf{u})|^2|\mathbf{D}(\mathbf{u}_n)|^2}{1+\frac{1}{n}|\mathbf{D}(\mathbf{u}_n)|^2}\,|v|\,d\mathbf{x}.\end{aligned}\tag{5.18}$$

Now we observe that by reasoning similarly as we did to prove (4.39)-(4.40), we also have

$$\mathbf{D}(\mathbf{u}_n) \to \mathbf{D}(\mathbf{u}) \quad \text{strongly in } \mathbf{L}^2(\Omega), \quad \text{as } n \to \infty, \tag{5.19}$$

$$\mathbf{D}(\mathbf{u}_n) \to \mathbf{D}(\mathbf{u}) \quad \text{a.e. in } \Omega, \quad \text{as } n \to \infty. \tag{5.20}$$

Thus, the last integral of (5.18) converges to zero by the application of Lebesgue's dominated convergence theorem, due to (5.20) and to assumption (2.19). With respect to the first of the two last integrals, we observe that by testing (4.58) with $\mathbf{v} = \mathbf{u}_n$, we have, due to the symmetry of $\mathbf{D}(\mathbf{u}_n)$, that

$$\int_\Omega (\nu + \nu_T(k_n))|\mathbf{D}(\mathbf{u}_n)|^2\, d\mathbf{x} = \int_\Omega \mathbf{g}\cdot\mathbf{u}_n\, d\mathbf{x} - \int_\Omega \mathbf{f}(\mathbf{u}_n)\cdot\mathbf{u}_n\, d\mathbf{x}. \tag{5.21}$$

Then, we take $\mathbf{v} = \mathbf{u}$ in (2.6), which we already know, by the first part of this section, that is valid. As a consequence, we also have

$$\int_\Omega (\nu + \nu_T(k))\, |\mathbf{D}(\mathbf{u})|^2\, d\mathbf{x} = \int_\Omega \mathbf{g}\cdot\mathbf{u}\, d\mathbf{x} - \int_\Omega \mathbf{f}(\mathbf{u})\cdot\mathbf{u}\, d\mathbf{x}\,. \tag{5.22}$$

Now, subtracting (5.22) to (5.21), we obtain

$$\begin{gathered}\int_\Omega \left(\nu_T(k_n)|\mathbf{D}(\mathbf{u}_n)|^2 - \nu_T(k)|\mathbf{D}(\mathbf{u})|^2\right) d\mathbf{x} = B_1 + B_2 + B_3 + B_4 := \\ \nu\int_\Omega \left(|\mathbf{D}(\mathbf{u})|^2 - |\mathbf{D}(\mathbf{u}_n)|^2\right) d\mathbf{x} + \int_\Omega (\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{u}_n))\cdot\mathbf{u}\, d\mathbf{x} + \\ \int_\Omega \mathbf{f}(\mathbf{u}_n)\cdot(\mathbf{u}-\mathbf{u}_n)\, d\mathbf{x} + \int_\Omega \mathbf{g}\cdot(\mathbf{u}_n - \mathbf{u})\, d\mathbf{x}\,.\end{gathered}$$

As $n \to \infty$, $B_1 \to 0$ due to (5.19). The terms B_2, B_3 and B_4 tend to zero, as n goes to infinity, by using in: B_2, (5.4) and a similar reasoning as we have used at (4.26); B_3, (2.8), (5.2) and (5.4); B_4, (3.1) and (5.4). Thence

$$\nu_T(k_n)|\mathbf{D}(\mathbf{u}_n)|^2 \to \nu_T(k)|\mathbf{D}(\mathbf{u})|^2 \quad \text{strongly in } \mathbf{L}^1(\Omega), \quad \text{as } m \to \infty$$

and, consequently, the first integral of the right-hand side of (5.18) also converges to zero.

Finally, we can pass to the limit $n \to \infty$ the equations (4.58)-(4.59) to obtain (2.6)-(2.7) for any $(\mathbf{v}, \varphi) \in \mathbf{V}\times \mathrm{W}_0^{1,q'}(\Omega)$.

The proof of Theorem 3.1 is now concluded. □

6. Uniqueness

In this section, we are interested to know in what conditions an uniqueness result for the turbulent problem (1.12)-(1.15) can be established. In the mathematical analysis of boundary values problems with feedback terms, there usually is a set of conditions related with the monotonicity of these terms that we also assume here,

$$(\mathbf{f}(\mathbf{u}_1) - \mathbf{f}(\mathbf{u}_2))\cdot(\mathbf{u}_1 - \mathbf{u}_2) \geq 0 \quad \text{for all } \mathbf{u}_1,\ \mathbf{u}_2 \in \mathbb{R}^d \quad \text{and} \quad \text{a.e. in } \Omega, \tag{6.1}$$

$$(\varepsilon(k_1) - \varepsilon(k_2))\,(k_1 - k_2) \geq 0 \quad \text{for all } k_1,\ k_2 \in \mathbb{R} \quad \text{and} \quad \text{a.e. in } \Omega. \tag{6.2}$$

The continuity conditions of the turbulent diffusion ν_D and of the turbulent viscosity ν_T on k, underlying to the assumption (2.2), need to be strengthened, in this section, to Lipschitz-continuity. Therefore, here we assume the existence of positive constants L_{ν_T} and L_{ν_D} such that

$$|\nu_T(k_1) - \nu_T(k_2)| \leq L_{\nu_T}|k_1 - k_2| \quad \text{for all } k_1,\ k_2 \in \mathbb{R} \quad \text{and} \quad \text{a.e. in } \Omega, \tag{6.3}$$

$$|\nu_D(k_1) - \nu_D(k_2)| \leq L_{\nu_D}|k_1 - k_2| \quad \text{for all } k_1,\ k_2 \in \mathbb{R} \quad \text{and} \quad \text{a.e. in } \Omega. \tag{6.4}$$

A Lipschitz condition on the continuity of the scalar field $\pi(\mathbf{u})$ or $\varpi(u)$ is also needed,

$$|\pi(\mathbf{u}_1) - \pi(\mathbf{u}_2)| \leq L_\pi\, |\mathbf{u}_1 - \mathbf{u}_2| \quad \text{for all } \mathbf{u}_1,\ \mathbf{u}_2 \in \mathbb{R}^d \quad \text{and} \quad \text{a.e. in } \Omega, \tag{6.5}$$

$$|\varpi(\mathbf{u}_1) - \pi(\mathbf{u}_2)| \leq L_\varpi\, |\mathbf{u}_1 - \mathbf{u}_2| \quad \text{for all } \mathbf{u}_1,\ \mathbf{u}_2 \in \mathbb{R}^d \quad \text{and} \quad \text{a.e. in } \Omega, \tag{6.6}$$

for some positive constants L_π and L_ϖ.

In the next result, we establish the uniqueness of weak solutions to the problem (1.12)-(1.15) under smallness assumptions on the solutions gradients $\|\nabla\mathbf{u}\|_{\mathbf{L}^\sigma(\Omega)}$ and $\|\nabla k\|_{\mathbf{L}^\tau(\Omega)}$, when we compare them, for particular values of σ and τ, with the kinematic viscosity ν and with the lower bound c_D of the turbulent diffusion ν_D (see (2.20)), respectively.

Theorem 6.1. *Let Ω be a bounded domain of $\mathbb{R}^d$, $2 \leq d \leq 4$, with a Lipschitz-continuous boundary $\partial\Omega$ and let $(\mathbf{u}, k)$ be a weak solution to the problem (1.12)-(1.15) in the conditions of Theorem 3.1. In addition, assume the conditions (6.1)-(6.2), (6.3)-(6.4) and (6.5)-(6.6) are fulfilled. If there exist positive constants $C_1 = C(\nu)$ and $C_2 = C(\nu, c_D)$ such that*

$$\|\nabla \mathbf{u}\|_{\mathbf{L}^\sigma(\Omega)} < C_1 \quad \textit{for some } \sigma > 2 \textit{ if } d = 2, \quad \textit{or for some } \sigma \geq d \textit{ if } d \neq 2, \tag{6.7}$$

$$\|\nabla k\|_{\mathbf{L}^\tau(\Omega)} < C_2 \quad \textit{for some } \tau > 2 \textit{ if } d = 2, \quad \textit{or for some } \tau \geq d \textit{ if } d \neq 2, \tag{6.8}$$

then the weak solution $(\mathbf{u}, k)$ is unique.

Recall that ν is the positive constant that accounts for the kinematic viscosity and c_D is another positive constant that accounts for the lower bound of the turbulent diffusion function (see (2.20)). The question of knowing the conditions under which (6.7) and (6.8) are satisfied, by the solutions to the problem (1.12)-(1.15), is being currently investigated by the authors. However, the case of $d \neq 2$ in (6.7)-(6.8) seems to be very difficult to achieve, which ultimately makes Theorem 6.1 as a 2-dimensional result. On the other hand, the cases of (6.7) and (6.8) for some $\sigma > 2$ and some $\tau > \frac{d}{d-1}$ regardless of the dimension, is an expectable result that we will show elsewhere.

Proof. Let $(\mathbf{u}_1, k_1)$ and $(\mathbf{u}_2, k_2)$ be two solutions of the problem (1.12)-(1.15) in the sense of Definition 2.1. We start by testing the equation (2.6) with $\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2$ and then we subtract the resulting equation when $\mathbf{u} = \mathbf{u}_2$ to the resulting equation when $\mathbf{u} = \mathbf{u}_1$. After some algebraic handling and using the assumptions (2.19) and (6.1) together with Korn's inequality, we obtain

$$\begin{aligned} \nu C_K^2 \int_\Omega |\nabla(\mathbf{u}_1 - \mathbf{u}_2)|^2 \, d\mathbf{x} \leq & - \int_\Omega (\nu_T(k_1) - \nu_T(k_2)) \mathbf{D}(\mathbf{u}_2) : \nabla(\mathbf{u}_1 - \mathbf{u}_2) \, d\mathbf{x} \\ & - \int_\Omega ((\mathbf{u}_1 - \mathbf{u}_2) \cdot \nabla) \, \mathbf{u}_2 \cdot (\mathbf{u}_1 - \mathbf{u}_2) \, d\mathbf{x} := I_1 + I_2, \end{aligned} \tag{6.9}$$

where C_K is the Korn's inequality constant. To estimate the term I_1, we use Hölder's inequality and Sobolev's inequality (2.26), together with the assumptions (6.3) and (6.7),

$$\begin{aligned} I_1 & \leq L_{\nu_T} \|k_1 - k_2\|_{\mathrm{L}^{2^*}(\Omega)} \|\nabla \mathbf{u}_2\|_{\mathbf{L}^\sigma(\Omega)} \|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|_{\mathbf{L}^2(\Omega)} \\ & \leq C_{I_1} \|\nabla(k_1 - k_2)\|_{\mathbf{L}^2(\Omega)} \|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|_{\mathbf{L}^2(\Omega)}, \quad C_{I_1} = L_{\nu_T} \lambda(2, d) C_1 \, . \end{aligned}$$

For I_2, we also use Hölder's inequality and now the vectorial version of the Sobolev inequality (2.26), to obtain

$$I_2 \leq \|\mathbf{u}_1 - \mathbf{u}_2\|^2_{\mathbf{L}^{2^*}(\Omega)} \|\nabla \mathbf{u}_2\|_{\mathbf{L}^\sigma(\Omega)} \leq C_{I_2} \|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|^2_{\mathbf{L}^2(\Omega)}, \quad C_{I_2} = \Lambda(2, d)^2 C_1 \, .$$

Now, gathering the estimates of I_1 and I_2 in (6.9), we obtain, after the use of Cauchy's inequality with a suitable ϵ,

$$\|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|^2_{\mathbf{L}^2(\Omega)} \leq C_I \|\nabla(k_1 - k_2)\|^2_{\mathbf{L}^2(\Omega)}, \quad C_I = \frac{C_{I_1}^2}{2\nu C_K^2} \left(\frac{\nu C_K^2}{2} - C_{I_2} \right)^{-1} . \tag{6.10}$$

Observe that this is possible as long as $\frac{\nu C_K^2}{2} - C_{I_2} > 0$, which, in view of (6.7), holds true by considering there $0 < C_1 < \frac{\nu C_K^2}{2\Lambda(2,d)^2}$.

Next, we test the equation (2.7) with $\varphi = k_1 - k_2$ and then we subtract the resulting equation when $k = k_2$ to the resulting equation when $k = k_1$. After some simplifications and using the assumptions (2.20) and (6.2), we obtain

$$\begin{aligned} & c_D \int_\Omega |\nabla(k_1 - k_2)|^2 \, d\mathbf{x} \leq \\ & - \int_\Omega (\mathbf{u}_1 \cdot \nabla k_1 - \mathbf{u}_2 \cdot \nabla k_2)(k_1 - k_2) \, d\mathbf{x} - \int_\Omega (\nu_D(k_1) - \nu_D(k_2)) \nabla k_2 \cdot \nabla(k_1 - k_2) \, d\mathbf{x} \\ & + \int_\Omega (\nu_T(k_1) |\mathbf{D}(\mathbf{u}_1)|^2 - \nu_T(k_2) |\mathbf{D}(\mathbf{u}_2)|^2)(k_1 - k_2) \, d\mathbf{x} + \int_\Omega (P(\mathbf{u}_1, k_1) - P(\mathbf{u}_2, k_2)) \, (k_1 - k_2) \, d\mathbf{x} \\ & := J_1 + J_2 + J_3 + J_4. \end{aligned} \tag{6.11}$$

After a simplification of J_1, we use Hölder's inequality and the scalar and vectorial versions of the Sobolev inequalities (2.24) and (2.26), to get

$$\begin{aligned} J_1 =& -\int_\Omega (\mathbf{u}_1\cdot(\nabla k_1-\nabla k_2)(k_1-k_2)\,d\mathbf{x}-\int_\Omega(\mathbf{u}_1-\mathbf{u}_2)\cdot\nabla k_2(k_1-k_2)\,d\mathbf{x}\\ \le& \|\mathbf{u}_1\|_{\mathbf{L}^\sigma(\Omega)}\|\nabla(k_1-k_2)\|_{\mathbf{L}^2(\Omega)}\|k_1-k_2\|_{\mathrm{L}^{2^*}(\Omega)}+\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbf{L}^{2^*}(\Omega)}\|\nabla k_2\|_{\mathbf{L}^\tau(\Omega)}\|\nabla(k_1-k_2)\|_{\mathbf{L}^2(\Omega)}\\ \le& C_{J_1,1}\|\nabla(k_1-k_2)\|^2_{\mathbf{L}^2(\Omega)}+C_{J_1,2}\|\nabla(\mathbf{u}_1-\mathbf{u}_2)\|_{\mathbf{L}^2(\Omega)}\|\nabla(k_1-k_2)\|_{\mathbf{L}^2(\Omega)}, \end{aligned}$$

where $C_{J_1,1}=\Lambda(\sigma,d)\lambda(2,d)C_1$ and $C_{J_1,2}=\Lambda(2,d)C_2$. Then, using the estimate (6.10), we obtain

$$J_1\le C_{J_1}\|\nabla(k_1-k_2)\|^2_{\mathbf{L}^2(\Omega)},\quad C_{J_1}=C_{J_1,1}+C_{J_1,2}\sqrt{C_I}.$$

As for the term J_2, we use the assumption (6.4) together with Hölder's inequality and Sobolev's inequality (2.26) in the following way,

$$\begin{aligned} J_2\le& L_{\nu_D}\|k_1-k_2\|_{\mathrm{L}^{2^*}(\Omega)}\|\nabla k_2\|_{\mathbf{L}^\tau(\Omega)}\|\nabla(k_1-k_2)\|_{\mathbf{L}^2(\Omega)}\\ \le& C_{J_2}\|\nabla(k_1-k_2)\|^2_{\mathbf{L}^2(\Omega)},\quad C_{J_2}=L_{\nu_D}\lambda(2,d)C_2. \end{aligned}$$

The term J_3 is firstly simplified and then we use the assumptions (2.19) and (6.3) together with Hölder's inequality, Sobolev's inequality (2.26) and Minkowski's inequality,

$$\begin{aligned} J_3 =& \int_\Omega(\nu_T(k_1)-\nu_T(k_2))\,|\mathbf{D}(\mathbf{u}_1)|^2(k_1-k_2)\,d\mathbf{x}+\int_\Omega\nu_T(k_2)\left(|\mathbf{D}(\mathbf{u}_1)|^2-|\mathbf{D}(\mathbf{u}_2)|^2\right)(k_1-k_2)\,d\mathbf{x}\\ \le& L_{\nu_T}\|k_1-k_2\|^2_{\mathrm{L}^{2^*}(\Omega)}\|\nabla\mathbf{u}_1\|^2_{\mathbf{L}^\sigma(\Omega)}+C_T\|k_1-k_2\|_{\mathrm{L}^{2^*}(\Omega)}\|\nabla(\mathbf{u}_1-\mathbf{u}_2)\|_{\mathbf{L}^2(\Omega)}\|\nabla\mathbf{u}_1+\nabla\mathbf{u}_2\|_{\mathbf{L}^\sigma(\Omega)}\\ \le& C_{J_3,1}\|\nabla(k_1-k_2)\|^2_{\mathbf{L}^2(\Omega)}+C_{J_3,2}\|\nabla(k_1-k_2)\|_{\mathbf{L}^2(\Omega)}\|\nabla(\mathbf{u}_1-\mathbf{u}_2)\|_{\mathbf{L}^2(\Omega)}, \end{aligned}$$

where $C_{J_3,1}=L_{\nu_T}\lambda(2,d)^2C_1^2$ and $C_{J_3,2}=C_T\lambda(2,d)2C_2$. Then, using (6.10), we obtain

$$J_3\le C_{J_3}\|\nabla(k_1-k_2)\|^2_{\mathbf{L}^2(\Omega)}\quad C_{J_3}=C_{J_3,1}+C_{J_3,2}\sqrt{C_I}\,.$$

With respect to the term J_4, we need to split the proof into the two possibilities: $P(\mathbf{u},k)=\pi(\mathbf{u})$ or $P(\mathbf{u},k)=\varpi(\mathbf{u})k$.

6.1. **If $P(\mathbf{u},k)=\pi(\mathbf{u})$ a.e. in Ω,** we use the assumption (6.5), Hölder's inequality and the scalar and vectorial versions of the Sobolev inequality (2.26), together with the estimate (6.10),

$$\begin{aligned} J_4 =& \int_\Omega(\pi(\mathbf{u}_1)-\pi(\mathbf{u}_2))\,(k_1-k_2)\,dx\le L_\pi\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbf{L}^{2^*}(\Omega)}\|k_1-k_2\|_{\mathrm{L}^{2^*}(\Omega)}\\ \le& C^\pi_{J_4}\|\nabla(k_1-k_2)\|^2_{\mathbf{L}^2(\Omega)},\quad C^\pi_{J_4}=L_\pi\Lambda(2,d)\lambda(2,d)\sqrt{C_I}, \end{aligned}$$

Where C_I is defined at (6.10).

6.2. **If $P(\mathbf{u},k)=\varpi(\mathbf{u})k$ a.e. in Ω,** we start by simplifying the writing of this term. Then, we use the assumption (6.6) together with Hölder's inequality and with the scalar and vectorial versions of the Sobolev inequality (2.26). We conjugate this with the application of the assumption (2.14) and with the estimate (6.10). After all, we obtain

$$\begin{aligned} J_4 =& \int_\Omega(\varpi(\mathbf{u}_1)-\varpi(\mathbf{u}_2))\,k_1(k_1-k_2)\,dx+\int_\Omega\varpi(\mathbf{u}_2)(k_1-k_2)^2\,dx\\ \le& L_\varpi\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbf{L}^{2^*}(\Omega)}\|k_1\|_{\mathrm{L}^{\tau^*}(\Omega)}\|k_1-k_2\|_{\mathrm{L}^{2^*}(\Omega)}+C_\varpi\|\mathbf{u}_2\|^\beta_{\mathbf{L}^{2^*}(\Omega)}\|k_1-k_2\|^2_{\mathrm{L}^{2^*}(\Omega)}\\ \le& C^\varpi_{J_4}\|\nabla(k_1-k_2)\|^2_{\mathbf{L}^2(\Omega)},\quad C^\varpi_{J_4}=L_\pi\Lambda(2,d)\lambda(\tau,d)\lambda(2,d)\sqrt{C_I}C_2+C_\varpi\Lambda(2,d)^\beta C_1^\beta, \end{aligned}$$

where C_I is defined at (6.10).

Now, gathering the estimates of J_1, J_2, J_3 and J_4 in (6.11), we obtain

$$(c_D-C_J)\int_\Omega|\nabla(k_1-k_2)|^2\,d\mathbf{x}\le 0,\quad C_J=\sum_{i=1}^4 C_{J_i},\tag{6.12}$$

where it should be noted that C_J depends on C_1 and C_2 and therefore depends on ν and c_D. As a consequence, it follows, by Sobolev's inequality, that $k_1=k_2$ a.e. in Ω, as long as we may choose C_1 and C_2 in (6.7)-(6.8) in such a way that $0<C_1<\frac{\nu C_K^2}{2\Lambda(2,d)^2}$ and $0<C_J<c_D$. Then, using (6.12) in (6.10), we can also infer that $\mathbf{u}_1=\mathbf{u}_2$ a.e. in Ω. □

7. Existence of a unique pressure

In this section, we will prove the existence and uniqueness of the last unknown to the problem (1.12)-(1.15) – the pressure. To establish this result, we will need to bound the gradients of the solutions $\mathbf{u}$ and k.

Proposition 7.1. *Let Ω be a bounded domain of $\mathbb{R}^d$, $2 \leq d \leq 4$, with a Lipschitz-continuous boundary $\partial\Omega$ and let $(\mathbf{u}, k)$ be a weak solution to the problem (1.12)-(1.15) in the conditions of Theorem 3.1. Then there exist positive constants C, C_1 and C_2 such that*

$$\|\nabla \mathbf{u}\|_{\mathbf{L}^2(\Omega)} \leq C\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}, \tag{7.1}$$

$$\|\nabla k\|_{\mathbf{L}^q(\Omega)} \leq C_1\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)} + C_2. \tag{7.2}$$

Proof. The estimates (7.1)-(7.2) follow easily by taking the lim inf in (5.2) and in (5.12), the last if $P(\mathbf{u}, k) = \pi(\mathbf{u})$ a.e. in Ω, or in (5.17) if $P(\mathbf{u}, k) = \varpi(\mathbf{u})k$ a.e. in Ω. □

Theorem 7.1. *Let Ω be a bounded domain of $\mathbb{R}^d$, $2 \leq d \leq 4$, with a Lipschitz-continuous boundary $\partial\Omega$ and let $(\mathbf{u}, k)$ be a weak solution to the problem (1.12)-(1.15) in the conditions of Theorem 3.1. Then there exists a unique $p \in \mathrm{L}^{\eta'}(\Omega)$, with $\int_\Omega p\, d\mathbf{x} = 0$ and η satisfying to (7.8) below, such that*

$$\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\, d\mathbf{x} + \int_\Omega (\nu + \nu_T(k))\mathbf{D}(\mathbf{u}) : \nabla\mathbf{v}\, d\mathbf{x} + \int_\Omega \mathbf{f}(\mathbf{u})\cdot\mathbf{v}\, d\mathbf{x} - \int_\Omega \mathbf{g}\cdot\mathbf{v}\, d\mathbf{x} = \int_\Omega p\ \mathbf{div}\,\mathbf{v}\, d\mathbf{x} \tag{7.3}$$

for any $\mathbf{v} \in \mathbf{W}^{1,\eta}_0(\Omega)$. Moreover, there exist positive constants C_1, C_2, C_3 and C_4 such that

$$\|p\|_{\mathrm{L}^{\eta'}(\Omega)} \leq C_1\|\nabla\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + C_2\|\nabla\mathbf{u}\|_{\mathbf{L}^2(\Omega)} + C_3\|\nabla\mathbf{u}\|^\alpha_{\mathbf{L}^2(\Omega)} + C_4\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}. \tag{7.4}$$

The proof of Theorem 7.1 will be made by using the following variant of de Rham's lemma due to Bogovskiĭ and Pileckas (see *e.g.* [16]).

Lemma 7.1. *Let us consider an arbitrary $\eta : 1 < \eta < \infty$ and $\mathbf{v}' \in \left(\mathbf{W}^{1,\eta}_0(\Omega)\right)' = \mathbf{W}^{-1,\eta'}(\Omega)$. If*

$$< \mathbf{v}', \mathbf{v} >_{\mathbf{W}^{-1,\eta'}(\Omega)\times\mathbf{W}^{1,\eta}_0(\Omega)} = 0 \quad \forall\, \mathbf{v} \in \mathbf{V}^\eta, \tag{7.5}$$

where $\mathbf{V}^\eta :=$ closure of $\mathcal{V}$ in $\mathrm{W}^{1,\eta}(\Omega)$, then there exists a unique $p \in \mathrm{L}^{\eta'}(\Omega)$, with $\int_\Omega p\, d\mathbf{x} = 0$, such that

$$< \mathbf{v}', \mathbf{v} >_{\mathbf{W}^{-1,\eta'}(\Omega)\times\mathbf{W}^{1,\eta}_0(\Omega)} = \int_\Omega p\ \mathbf{div}\,\mathbf{v}\, d\mathbf{x}$$

for any $\mathbf{v} \in \mathbf{W}^{1,\eta}_0(\Omega)$. Moreover, there exists a positive constant C such that

$$\|p\|_{\mathrm{L}^{\eta'}(\Omega)} \leq C\|\mathbf{v}'\|_{\mathbf{W}^{-1,\eta'}(\Omega)}.$$

Proof. To prove Theorem 7.1, we start by defining

$$\mathbf{Q} := (\mathbf{u}\cdot\nabla)\mathbf{u} - \mathbf{div}((\nu + \nu_T(k))\mathbf{D}(\mathbf{u})) + \mathbf{f}(\mathbf{u}) - \mathbf{g} \tag{7.6}$$

and let us consider the linear operator $\mathbf{v}'$ defined by

$$< \mathbf{v}', \mathbf{v} >_{\mathbf{W}^{-1,\eta'}(\Omega)\times\mathbf{W}^{1,\eta}_0(\Omega)} = \int_\Omega \mathbf{Q}\cdot\mathbf{v}\, d\mathbf{x}\,. \tag{7.7}$$

It can be proved that $\mathbf{v}' \in \mathbf{W}^{-1,\eta'}(\Omega)$ for

$$\eta \geq \max\left\{2, \frac{d}{2}, \frac{2d}{2d - \alpha(d-2) + 2}\right\} \text{ if } d \neq 2, \quad \text{or} \quad \eta \geq 2 \text{ if } d = 2. \tag{7.8}$$

In fact, we have

$$\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{\mathbf{W}^{-1,\eta'}(\Omega)} \leq \|\mathbf{u}\|^2_{\mathbf{L}^{2\eta'}(\Omega)} \leq C_1\|\nabla\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} \quad \text{for} \quad \eta \geq \frac{d}{2}, \tag{7.9}$$

$$\|\mathbf{div}((\nu + \nu_T(k))\mathbf{D}(\mathbf{u}))\|_{\mathbf{W}^{-1,\eta'}(\Omega)} \leq C_T\|\nabla\mathbf{u}\|_{\mathbf{L}^{\eta'}(\Omega)} \leq C_2\|\nabla\mathbf{u}\|_{\mathbf{L}^2(\Omega)} \quad \text{for} \quad \eta \geq 2, \tag{7.10}$$

$$\|\mathbf{f}(\mathbf{u})\|_{\mathbf{W}^{-1,\eta'}(\Omega)} \leq \sup_{\|\varphi\|_{\mathbf{W}^{1,\eta}_0(\Omega)}=1} C_f\|\mathbf{u}\|^\alpha_{\mathbf{L}^{2^*}(\Omega)}\|\varphi\|_{\mathbf{L}^{\eta^*}(\Omega)} \leq C_3\|\nabla\mathbf{u}\|^\alpha_{\mathbf{L}^2(\Omega)} \quad \text{for} \quad \eta \geq \frac{2d}{2d - \alpha(d-2) + 2}, \tag{7.11}$$

$$\|\mathbf{g}\|_{\mathbf{W}^{-1,\eta'}(\Omega)} \leq \|\mathbf{g}\|_{\mathbf{L}^{\eta'}(\Omega)} \leq C_4\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)} \quad \text{for} \quad \eta \geq 2, \tag{7.12}$$

where $C_1 = C(\eta, d, \Omega)$, $C_2 = C(C_T, \eta, \Omega)$, $C_3 = C(\alpha, C_f, \eta, d, \Omega)$ and $C_4 = C(\eta, \Omega)$ are positive constants. Observe that in (7.10) we have used (2.19) and in (7.11) we used (2.8). Note also that, in (7.11), the assumption (2.8) implies that $2d - \alpha(d-2) + 2 > 0$ for $d \neq 2$. For $d = 2$, (7.11) holds for any $\eta \geq 1$.

On the other hand, since $\eta \geq \frac{d}{2}$, the Sobolev imbedding $\mathbf{W}_0^{1,\eta}(\Omega) \hookrightarrow \mathbf{L}^d(\Omega)$ holds. As a consequence, and in view of (2.6), we can infer that (7.5), with $\mathbf{v}'$ defined through (7.7), holds. Finally (7.3) follows from Lemma 7.1 and, in particular, (7.4) is a consequence of (7.9)-(7.12). □

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