THE FRANK TENSOR AS BOUNDARY CONDITION IN INTRINSIC LINEARIZED ELASTICITY

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Abstract. The Frank tensor plays a crucial role in linear elasticity, and in particular in the presence of dislocation lines, since its curl is exactly the elastic strain incompatibility. Furthermore, the Frank tensor also appears in Cesaro decomposition, and in Volterra theory of dislocations and disclinations, since its jump is the Frank vector around the defect line. The purpose of this paper is to show to which functional space the compatible strain e belongs in order to imply a homogeneous boundary conditions for the induced displacement field on a portion Γ_0 of the boundary. This will allow one to define the homogeneous, or even the mixed problem of linearized elasticity in a variational setting involving the strain e in place of displacement u. With other puposes, this problem was originaly treated by Ph. Ciarlet and C. Mardare, and termed the *intrinsic formulation*. In this paper we propose alternative conditions on e expressed in terms of e and the Frank tensor $\mathrm{Curl}^{\mathrm{t}}\, e$ only, yielding a clear physical understanding and showing as equivalent to Ciarlet-Mardare boundary condition.

1. Introduction

1.1. **The intrinsic formulation of elasticity.** One the one hand, Pysicists and mechanical Engineers mostly consider strain and stress as their basic model variable in Elasticity, both for theoretical and computational reasons. Indeed, given the stress tensor, the strain is then well defined as soon as a constitutive law is provided, here a linear homogeneous and isotropic law. Thus strain-stress constitutive law reads $\epsilon = \mathbb{C}\sigma$, with $\mathbb{C}$ the compliance tensor, i.e., fourth-rank (inverse) tensor of elasticity.

On the other hand, Mathematicians working in Elasticity, prefer the displacement as model field, from which the strain is defined by the kinematic relation $\epsilon = \nabla^S u$, and then the stress by a constitutive law. This choice presumably comes from the study of elliptic boundary-value problems, where the elasticity system is seen as a vector-valued variable extension of the elliptic equations in divergence form. Moreover, weak and variational formulations are most easily derived by means of the displacment, and show a convenient and elegant way of solving problems in Elasticity.

There are profound theoretical arguments to refrain from taking the displacement as main model variable. For instance, its possible multivaluedness, which is not to avoid from a Physical standpoint, since multivaluedness may have a meaning, but which must be addressed in an adequate manner in the chosen mathematical formalism. Another example is the reference configuration issue: while natural in finite elasticity as soon as an undeformed body is defined, it becomes difficult to aprehend in linearized elasticity, since the deformed and undeformed configurations are said to coincide, thus questioning the definition of the displacement field as a variation between the current and the reference positions. Let us also mention plasticity or defective bodies, where stress and defect-free reference configurations might not exist (simultaneously, as intended), not to mention the possible use of intermediate configurations, which induce a plastic and an elastic deformation (in whatever favorite order, but as such, depending on the choice of this intermediate configuration), whose Physical meaning is far from clear. In fact, what is a plastic distortion (i.e., the gradient of a plastic displacement) if no constitutive law exist for the rotations (i.e., its skewsymmetric part)? Not to mention the fact that in principle any rigourous model should be proven independent of the choice any reference configuration.

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For these reasons, the intrinsic approach in linearized elasticity by Ph. Ciarlet and C. Mardare [5] constitutes an extremely valuable attempt to reconcile in an elegant manner the two aforementioned approaches and Scientific communities. In their presentation, the strain is the main model variable, while variational formulations are sought. The displacement only appears in a second step if the Riemannian curvature tensor associated to the elastic metric vanishes (for the use of differential geoemtry concepts in Elasticity with defects, see e.g. [9, 10, 18]). These strains are said compatible, as it is immediately remarked [14] that the above geometric differential condition is equivalent, to the first order, to requiring that the strain has vanishing incompatibility. In Ciarlet and Mardare approach a differential geometric setting is chosen (see also [3]), where the boundary under analysis is defined by means of smooth enough immersions, from which the curvilineat basis, the metric, the symmetric connexion and the curvature tensors are derived. It should be emphasized that such derived curvilinear basis are indeed defined in the body itself as well as on its boundary, but are not mutually orthogonal, and have no particular Physical meaning.

In contrast, in the proposed approach our wish is to define orthonormal basis in the body, which have a Physical meaning, that is, are in some sense intrinsic to the body, while also considering an approach of differential geometry in a first step. Only in a second step, the appropriate functional spaces with Hilbertian structure are introduced and proved well suited to study the Elasticity problem in intrinsic terms. Moreover, since our curvilinear basis is chosen orthonormal the associated metric (in the body and on its surface) reduces to the identity. In particular, the notions of covariant and contravariant components of a vector/tensor do coincide.

Our model variables will be the strain end the Frank tensor, that is the transpose of the strain curl, which also bear a clear physical meaning. Our aim is thus to determine a well-posed variational problem in terms of the strain only and with prescribed conditions of these two quantities on a connected subset Γ_0 of the boundary. In our approach, the displacement also appears in a second step, as one of a two-field strain decomposition, and is to be considered, by this procedure, as a mere mathematical object, whose interpretation as displacement is made for conveniance, and obvisously coincide with the classical displacement field in the special case of compatible strains.

1.2. **Notations and conventions.** Let Ω be a bounded domain of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, i.e., the body Ω is embedded in a Euclidean manifold. By smooth we mean C^∞, but this assumption could be considerably weakened. Let $\mathbb{M}^3$ denote the space of square 3-matrices, and $\mathbb{S}^3$ of symmetric 3-matrices. Curl, incompatibility and cross product with second-rank tensors are defined componentwise as follows with the summation convention on repeated indices. Let the Cartesian base be denoted by $\{e^i\}$ and the associated Cartesian coordinates by x_i. In the following definiton, E represents a second-rank tensor, N is a unit vector (which will be extended from the boundary to the domain), and ϵ is the Levi-Civita symbol. Furthermore ∂_{x_i} stands for partial derivative with respect to x_i (further in the paper ∂_i will mean a curvilinear derivative). In the Cartesian basis, one has:

$$\begin{aligned} (\,\mathrm{Curl}\ E)_{ij} &:= (\nabla\times E)_{ij} = \epsilon_{jkm}\partial_{x_k}E_{im}, \\ (\,\mathrm{inc}\ E)_{ij} &:= (\,\mathrm{Curl}\ \ \mathrm{Curl}^{\mathrm{t}}\ E)_{ij} = \epsilon_{ikm}\epsilon_{jln}\partial_{x_k}\partial_{x_l}E_{mn}, \\ (N\times E)_{ij} &:= -(E\times N)_{ij} = \epsilon_{jkm}N_kE_{im}. \end{aligned}$$

Note that the expression of the incompatibility in a general curvilinear basis is a difficult issue addressed in [19].

1.3. **Origin of our approach: the Frank tensor and Cesaro-Volterra identities.** As a first step, let us recall the problem of reconstructing a displacement from a given symmetric tensor (see, e.g., [13] for this classical topic). In linearized elasticity, if all the functions involved are smooth enough, we prove that the displacement field u is completely defined in terms of the linearized strain tensor e by a recursive integral formula (cf. (1.8)), which we compute explicitly.

Let $e \in C^\infty(\Omega, \mathbb{M}^3)$ be a symmetric tensor field such that inc $e = 0$ on Ω. Let us fix $x_0, x \in \Omega$, and let $\gamma \in C^1([0,1],\Omega)$ be a curve in Ω such that $\gamma(0) = x_0$ and $\gamma(1) = x$. We define the following quantities:

$$w_i(x;\gamma) \;:=\; w_i(x_0) + \int_\gamma \epsilon_{ipn}\partial_p e_{mn}(y)\mathrm{d}y_m \tag{1.1}$$

$$u_i(x;\gamma) \;:=\; u_i(x_0) + \int_\gamma \left(e_{il}(y) - \epsilon_{ilk}w_k(y)\right) dy_l. \tag{1.2}$$

Let us now prove that the quantities $w(x)$ and $u(x)$ defined in (1.1) and (1.2) do not depend on the choice of the path from x_0 to x. We will show that this is a consequence of the fact that inc e = 0. In such a case the quantities w and u define two C^∞ functions on Ω that will be called the rotation and the displacement vectors associated to the strain e, respectively. In order to prove this fact, we compute the jump of w and u between two arbitrary curves with the same endpoints, and observe that this quantity is zero if and only if the incompatibility tensor vanishes. These are exactly the well known Saint-Venant compatibility relations.

The rotation and displacement jumps are defined as

$$[\![w_i]\!](x;x_0) := w_i(x;\gamma) - w_i(x;\tilde\gamma), \tag{1.3}$$

$$[\![u_i]\!](x;x_0) := u_i(x;\gamma) - u_i(x;\tilde\gamma), \tag{1.4}$$

respectively. Here, γ and $\tilde\gamma$ are two distinct curves with start- and endpoints x_0 and x. Therefore the jumps will be non-vanishing as soon as $\gamma - \tilde\gamma$ encloses at least ones a dislocation line, by Stokes theorem (cf. [14]). In particular $[\![w_i]\!]$ defines the Frank vector, associated to the Frank tensor by (1.1), while $[\![u_i]\!]$ defines the Burgers vector, associated to the Frank tensor by (1.2).

Theorem 1.1 (Rotation and displacement jumps [14])**.** *Let $\Omega \subseteq \mathbb{R}^3$ be a simply-connected domain, let $x_0 \in \Omega$ be prescribed, and let $w, u \in C^\infty(\Omega, \mathbb{R}^3)$ be the functions defined in* (1.1) *and* (1.2)*, respectively. Then the following formulae hold:*

$$[\![w_i]\!](x;x_0) = \int_{S_{\gamma-\tilde\gamma}} (\mathrm{inc}\ e(y))_{im}\mathrm{d}S_m(y), \tag{1.5}$$

$$[\![u_i]\!](x;x_0) = \int_{S_{\gamma-\tilde\gamma}} (y_m - x_m)\epsilon_{imk}(\mathrm{inc}\ e(y))_{qk}\mathrm{d}S_q(y), \tag{1.6}$$

for all $x \in \Omega$, and where $S_{\gamma-\tilde\gamma}$ is a surface enclosed by the the closed path $\gamma - \tilde\gamma$. In particular,

$$[\![w_i]\!], [\![u_i]\!] = 0 \textit{ for each couple of curves } \gamma,\ \tilde\gamma \iff \mathrm{inc}\ e = 0.$$

Remark 1.2. *As a consequence of* inc $e = 0$, (1.1) *and* (1.2) *do not depend on the choice of the curve $\gamma \in \mathscr{C}^1([0,1],\Omega)$ connecting x_0 to x. In particular, the vector fields $w \in \mathcal{C}^\infty(\Omega,\mathbb{R}^3)$ and $u \in \mathcal{C}^\infty(\Omega,\mathbb{R}^3)$ are univoquely defined. Thus, in* (1.1) *and* (1.2)*, one can use the notation* $\displaystyle\int_\gamma = \int_{x_0}^x$.

It is straightforward to prove the following result:

Corollary 1.3 (Saint-Venant compatibility conditions in C^∞;[14])**.** *Let Ω be a simply-connected and bounded open set in $\mathbb{R}^3$ and let $e \in C^\infty(\Omega, \mathbb{M}^3)$ be a symmetric tensor field. Then there exists $u \in C^\infty(\Omega,\mathbb{R}^3)$ (given by* (1.2)*) such that $e = \nabla^s u$, if and only if*

$$\mathrm{inc}\ e = 0. \tag{1.7}$$

Now, the following classical quantities can be introduced:

Definition 1.4 (Strain and Frank tensor)**.** *Let $u : \Omega \to \mathbb{R}^3$ be a smooth displacement field, i.e., it writes in the Cartesian base $\{e^i\}$ as $u = \hat u_i e^i$. Let us introduce the following quantities:*

(i) *the linear elatic strain $e_{ij} := (\nabla u)\cdot e^i \otimes e^j = \frac{1}{2}\left(\partial_{x_j}\hat u_i + \partial_{x_i}\hat u_j\right)$;*
(ii) *the Frank tensor $\epsilon_{ijpk}\partial_p e_{jk} = (\,\mathrm{Curl}^{\mathrm{t}}\ e)_{ij}$,*

where x_i stands for the ith Cartesian coordinate.

The term Frank tensor comes from the fact that its integration on a closed curve around a dislocation, yields by (1.1), the jump of the rotation vector, which is known as the Frank vector [11, 12].

Remark 1.5. *Let $x \in \Omega$ and γ_x be a smooth curve joining x_0 to x. Let $y \in \gamma_x$ and γ_y be a smooth curve joining x_0 to y. By Eqs* (1.1) *and* (1.2)*, the displacement writes as a recursive line integral involving the strain and the Frank tensor, i.e.,*

$$u_i(x;\gamma_x) = u_i(x_0) + \int_{\gamma_x} \left(e_{il}(y) - \epsilon_{ilk} \left(w_i(x_0) + \int_{\gamma_y} (\,\mathrm{Curl}^{\mathrm{t}}\ e)_{im}(\xi)\mathrm{d}\xi_m \right)(y) \right) dy_l. \tag{1.8}$$

1.4. **Motivation of our approach and purpose of the paper.** It is seen by (1.8) that the assumed smooth displacement in linearized elasticity can be expressed by means of the strain, here a general symmetric tensor e, in practice a tensor derived from the stress tensor by a constitutive law, and of the transpose of the curl of the strain, called the Frank tensor. On the one side, this classical formula is at the basis of the intrinsic, strain-based formulation of elasticity, on the other, it directly provided the Frank and Burgers vectors in terms of the strain and its curl. Hence, it appears that the Frank tensor plays a crucial role in elasticity with dislocations.

Based on these considerations, the purpose of this paper is to show to which functional space e belongs in order to imply a homogeneous boundary conditions in terms of the displacement field on a connected subset Γ_0 of $\partial\Omega$. This will allow one to define the homogeneous, or even the mixed problem of elasticity in a variational setting involving e in place of u. This problem was originaly treated by Ph. Ciarlet and C. Mardare, and termed the intrinsic approach [5], where they determined in differential geometric terms which conditions e shoud satisfy on the boundary. In this paper we present a alternative conditions on e expressed in terms of the Frank tensor only, thereby providing a Physical understanding of the boundary condition. We also show that it is equivalent to Ciarlet-Mardare condition. Moreover, we believe that this intrinsic approach is mandatory when dislocation lines are present, since the displacement, being multiple-valued by (1.2), is an uncomfortable model variable (a rigorous manner of introducing the displacment as a main variable, though, is for instance to consider torus-valued maps as done in [15]). In the presence of dislocations, the Frank tensor is a model variable, beside the elastic strain, and its curl is precisely the strain incompatibility as related to the dislocation density. For these reasons our belief is that it is worth modelling linearized elasticity in terms of these two tensors. In fact the classical setting will appear as soon as the dislocation density vanishes (and thus the strain incompatibility). This paper treats a particular such aspect as related to the relation between a strain/Frank tensor-based and a displacement-based boundary condition.

2. Extension and differentiation of the normal and tangent vectors to a surface

The aim here is to construct a curvilinear basis on the boundary which should be smooth and also orthonormal, starting from the vector $N_{\partial\Omega}$ normal to the boundary and defining two tangent vectors perpendicular to $N_{\partial\Omega}$. This basis is then extended to the whole body. The natural moving frame sought is close in spirit to the Darboux frame of surfaces, though in principle the latter may only be defined at non-umbilical points. As a matter of fact, in order to achieve a certain level of generality, we will not consider principal lines of curvature with their associated principal curvatures, and hence the gradient of the normal vector will be given by a symmetric matrix with possibly non-zero extradiagonal components.

2.1. **Signed distance function and extended unit normal.** We denote by $N_{\partial\Omega}$ the outward unit normal to $\partial\Omega$, and by b the signed distance to $\partial\Omega$, i.e.,

$$b(x) = \begin{cases} \ \ \mathrm{dist}(x,\partial\Omega) & \text{if } x \notin \Omega, \\ -\mathrm{dist}(x,\partial\Omega) & \text{if } x \in \Omega. \end{cases}$$

We recall the following result.

Theorem 2.1 ([7], Chap. 5, Thms 3.1 and 4.3)**.** *There exists an open neighborhood W of $\partial\Omega$ such that*

1. *b is smooth in W;*
2. *every $x \in W$ admits a unique projection $p_{\partial\Omega}(x)$ onto $\partial\Omega$;*

(3) *this projection satisfies* $p_{\partial\Omega}(x) = x - \frac{1}{2}\nabla b^2(x), \qquad x \in W$;
(4) *it holds* $\nabla b(x) = N_{\partial\Omega}(p_{\partial\Omega}(x)), \qquad x \in W$.

In particular, this latter property shows that $\nabla b(x) = N_{\partial\Omega}(x)$ for all $x \in \partial\Omega$ and $|\nabla b(x)| = 1$ for all $x \in W$. Therefore, we define the extended unit normal by

$$N(x) := \nabla b(x) = N_{\partial\Omega}(p_{\partial\Omega}(x)), \qquad x \in W. \tag{2.1}$$

2.2. **Tangent vectors and orthonormal frame on $\partial\Omega$.** For all $x \in \partial\Omega$, we denote by $T_{\partial\Omega}(x)$ the tangent plane to $\partial\Omega$ at x, that is, the orthogonal complement of $N_{\partial\Omega}(x)$. As $\partial\Omega$ is smooth, there exists a covering of $\partial\Omega$ by open balls $B_1, ..., B_M$ of $\mathbb{R}^3$ such that, for each index k, two smooth vector fields $\tau^A_{\partial\Omega}, \tau^B_{\partial\Omega}$ can be constructed on $\partial\Omega \cap B_k$ where, for all $x \in \partial\Omega \cap B_k$, $(\tau^A_{\partial\Omega}(x), \tau^B_{\partial\Omega}(x))$ is an orthonormal basis of $T_{\partial\Omega}(x)$. In all the sequel, the index k will be implicitly considered as fixed and the restriction to B_k will be omitted. In fact, for our needs, global properties and constructions will be easily obtained from local ones through a partition of unity subordinate to the covering.

Using that the Jacobian matrix $DN(x) = D^2b(x)$ of $N(x)$ is symmetric, differentiating the equality $|N(x)|^2 = 1$ entails $\partial_N N(x) = DN(x)N(x) = 0$, $x \in W$. In other words, $N(x)$ is an eigenvector of $DN(x)$ for the eigenvalue 0. For all $x \in \partial\Omega$, the system $(\tau^A_{\partial\Omega}(x), \tau^B_{\partial\Omega}(x), N_{\partial\Omega}(x))$ is an orthonormal basis of $\mathbb{R}^3$. In this basis, $DN(x)$ takes the form

$$DN(x) = \begin{pmatrix} \kappa^A_{\partial\Omega}(x) & \xi_{\partial\Omega}(x) & 0 \\ \xi_{\partial\Omega}(x) & \kappa^B_{\partial\Omega}(x) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad x \in \partial\Omega, \tag{2.2}$$

where $\kappa^A_{\partial\Omega}$, $\kappa^B_{\partial\Omega}$ and ξ are smooth scalar fields defined on $\partial\Omega$. If $R \in \{A, B\}$, we denote by R^* the complementary index of R, that is, $R^* = B$ if $R = A$ and $R^* = A$ if $R = B$.

2.3. **Extended tangent vectors and the parallel curvinormal frame.** Let d be defined in W by

$$d := \left(1 + b\ \kappa^A_{\partial\Omega} \circ p_{\partial\Omega}\right)\left(1 + b\ \kappa^B_{\partial\Omega} \circ p_{\partial\Omega}\right) - \left(b\ \xi_{\partial\Omega} \circ p_{\partial\Omega}\right)^2.$$

Possibly adjusting W so that $d(x) > 0$ for all $x \in W$, we define in W:

$$\tau^R = \tau^R_{\partial\Omega} \circ p_{\partial\Omega}, \quad \kappa^R = d^{-1}\left((1 + b\ \kappa^{R^*}_{\partial\Omega} \circ p_{\partial\Omega})(\kappa^R_{\partial\Omega} \circ p_{\partial\Omega}) - b\ (\xi_{\partial\Omega} \circ p_{\partial\Omega})^2\right), \tag{2.3}$$

$$\xi = d^{-1}\xi_{\partial\Omega} \circ p_{\partial\Omega}, \quad \kappa = \kappa^A + \kappa^B, \qquad \gamma^R = \operatorname{div} \tau^R. \tag{2.4}$$

Obviously, for each $x \in W$, the triple $\left(\tau^A(x), \tau^B(x), N(x)\right)$ forms an orthonormal basis of $\mathbb{R}^3$, which we call the the *curvinormal* (parallel) frame. Next, we compute the normal and tangential derivatives of these vectors. We denote the tangential derivative ∂_{τ^R} by ∂_R for simplicity, i.e., $\partial_R u := Du\tau^R$, where Du stands for the differential of u, and $\partial_R u$ its value in the direction τ^R.

Theorem 2.2 ([1]). *The following holds in W:*

$$\begin{aligned} \partial_N \tau^R &= 0, \partial_R N = \kappa^R \tau^R + \xi \tau^{R^*}, \partial_R \tau^R = -\kappa^R N - \gamma^{R^*}\tau^{R^*}, \partial_{R^*}\tau^R = \gamma^R \tau^{R^*} - \xi N, \\ \operatorname{div}\ N &= \operatorname{tr}\ DN = \Delta b = \kappa. \end{aligned} \tag{2.5}$$

In this paper we make use of a orthonormal frame parallel to the boundary. The induced coordinates are non-holonomic in the following sense.

Corollary 2.3 (Non-holonomic curvinormal frame [1]). *If f is twice differentiable in Ω it holds*

$$\partial_R \partial_N f - \partial_N \partial_R f = \kappa^R \partial_R f + \xi \partial_{R^*} f. \tag{2.6}$$

3. Differential geometry on the boundary with curvinormal basis

At each point $x \in \partial\Omega$ the curvinormal basis $\left(g^i(x)\right)_{i=A,B,N} := \left(\tau^A(x), \tau^B(x), N_{\partial\Omega}(x)\right)$ is orthonormal and differentiable by Theorem 2.2. Remark that indices P, Q, R will stand for A or B, and denote one of the two orthogonal tangent vectors on the boundary, whereas index N will always be associated to the normal $N_{\partial\Omega}$. In some sense, the chosen curvilinear basis is a generalization to general surfaces of the spherical or cylindrical basis. We recall that ∂_i means the differential in the direction g^i. Let u be a scalar. Then, $\partial_i u = \partial_R u = \tau^R \cdot \nabla u$ for $R = A, B$, or $\partial_N u = N \cdot \nabla u$ for $i = N$, with $\nabla = e^i \partial_{x_i}$ the Cartesian gradient operator, where

e^i stands for the ith Cartesian basis vector. Recall that partial curvilinear derivatives do not commute, as shown in Corollary 2.3. For instance, the gradient in spherical coordinates reads $\nabla u = \partial_r u e_r + \frac{1}{r}\partial_\phi u e_\Phi + \frac{1}{r\sin\phi}\partial_\theta u e_\theta$ and hence $\partial_A = \frac{1}{r}\partial_\phi$ and $\partial_B = \frac{1}{r\sin\phi}\partial_\theta$.

Furthermore, let $x = x_j e^j$ denote the position vector, and $g^i = g^i_k e^k$ be the ith curvilinear basis vector. Then by definition, $\partial_i x = g^i_k \partial_{x_k} x = g^i_k e^k = g^i$.

3.1. **Christoffel symbols and Riemannian curvature.** A vector such as the displacement field will write as $u = u_i g^i$ with u_i its covariant components. Moreover, the extrinsic metric is Euclidean, since $g^{ij} := g^i \cdot g^j = \delta^{ij}$. Let $g_i := g^{ij} g^j$ be the dual of the basis vector. The Christoffel symbol of second kind Γ^p_{ij} is defined as the linear operator such that [3]

$$\partial_j g^p = -\Gamma^p_{ij} g^i, \tag{3.1}$$

called the Levi-Civita connection. In other words,

$$\Gamma^p_{ij} := -g_i \cdot \partial_j g^p.$$

Remark that the body manifold in this paper is Euclidean and hence the associated connection is symmetric. However, the associated Christoffel symbols are not symmetric due to the choice of a non-holonomic frame, since $\Gamma^p_{ij} := -g_i \cdot \partial_j g^p = -\partial_j(g^p \cdot g_i) = g^p \cdot \partial_j g^i = g^p \cdot \partial_j \partial_i x$ where $\partial_j \partial_i x \neq \partial_j \partial_i x$ by our choice of an non-holonomic frame, see Lemma 2.3.

As a consequence of (3.1), it holds

$$\partial_j u = \partial_j(u_i g^i) = (\partial_j u_i - \Gamma^p_{ij} u_p) g^i = u_{i\|j} g^i, \tag{3.2}$$

where the covariant derivative of the covariant component of u reads

$$u_{i\|j} := \partial_j u_i - \Gamma^p_{ij} u_p. \tag{3.3}$$

By Theorem 2.2, it is easily deduced by identification with (3.1) that the only nonvanishing components of Γ^p_{ij} read (with no sum on repeated indices)

$$\Gamma^N_{RR^*} = -\xi,\ \Gamma^N_{RR} = -\kappa^R, \Gamma^{R^*}_{NR} = \xi,\ \Gamma^R_{R^*R} = \gamma^{R^*},\ \Gamma^R_{NR} = \kappa^R,\ \Gamma^R_{R^*R^*} = -\gamma^R. \tag{3.4}$$

As an example, in a spherical coordinates/components system, it holds[1] $i, j \in \{\phi, \theta\}$, $\kappa^R = \frac{1}{r}$, $\gamma^\phi = \frac{1}{\tan\phi}$, $\gamma^\theta = 0$, and hence

$$\Gamma^r_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{r} & 0 \\ 0 & 0 & -\frac{1}{r} \end{pmatrix},\ \Gamma^\phi_{ij} = \begin{pmatrix} 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{r\tan\phi} \end{pmatrix},\ \Gamma^\theta_{ij} = \begin{pmatrix} 0 & 0 & \frac{1}{r} \\ 0 & 0 & \frac{1}{r\tan\phi} \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.5}$$

Moreover, it is observed that Γ^p_{ij} is not symmetric, i.e., $\Gamma^p_{ij} \neq \Gamma^p_{ij}$. This Euclidean metric is therefore associated with a nonvanishing "anholonomicity torsion" (as opposed to the connection torsion, which here vanishes),

$$T^p_{ij} := \Gamma^p_{ij} - \Gamma^p_{ji}.$$

In the curvinormal basis, it is easily computed that the only nonvanishing components of T^p_{ij} are

$$T^R_{ij} = \kappa^R \delta_{iN}\delta_{jR} + \xi\delta_{iN}\delta_{jR^*} + (\gamma^{R^*} - \gamma^R)\delta_{iR^*}\delta_{jR}.$$

In particular in spherical coordinates one has

$$T^r_{ij} = 0,\ T^\phi_{ij} = \begin{pmatrix} 0 & \frac{1}{r} & 0 \\ -\frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ T^\theta_{ij} = \begin{pmatrix} 0 & 0 & \frac{1}{r} \\ 0 & 0 & \frac{1}{r\tan\phi} \\ -\frac{1}{r} & -\frac{1}{r\tan\phi} & 0 \end{pmatrix}. \tag{3.6}$$

The Riemann curvature tensor is defined as [8]

$$\mathrm{Riem}^q_{ijk} := \partial_k \Gamma^q_{ij} - \partial_j \Gamma^q_{ik} + \Gamma^p_{ij}\Gamma^q_{pk} - \Gamma^p_{ik}\Gamma^q_{pj}. \tag{3.7}$$

Accordingly, the Ricci curvature tensor is defined as its jq-trace, viz.,

$$\mathrm{Ric}_{ik} := \mathrm{Riem}^q_{iqk} := \partial_k \Gamma^q_{iq} - \partial_q \Gamma^q_{ik} + \Gamma^p_{iq}\Gamma^q_{pk} - \Gamma^p_{ik}\Gamma^q_{pq}. \tag{3.8}$$

Lastly the scalar curvature is the trace of Ric, i.e.,

$$\mathrm{R} := \mathrm{Ric}_{kk} := \partial_k \Gamma^q_{kq} - \partial_q \Gamma^q_{kk} + \Gamma^p_{kq}\Gamma^q_{pk} - \Gamma^p_{kk}\Gamma^q_{pq}. \tag{3.9}$$

[1]Here, ϕ denotes the polar, and θ the azimuthal coordinate, respectively.

In the curvinormal basis, it is easily computed from (3.9) and (3.4) that (with sum on R),

$$\mathrm{R} = 2\partial_N \kappa + 2\partial_R \gamma^R + 2\left(\xi^2 + (\gamma^R)^2\right), \tag{3.10}$$

which for a spherical surface or radius r (for which $\xi = 0$, $\gamma^R = \frac{1}{r\tan\phi}$ and $\kappa^R = 1/r$), yields $\mathrm{R} = 4/r^2 - 2/(r^2\sin^2\phi) + 2/(r^2\tan^2\phi) = 2/r^2$, that is twice the Gaussian curvature, as expected.

Remark 3.1. *Let us emphasize that Eq.* (3.10) *yields a relation between the scalar curvature and the normal derivative of the mean curvature $H = \kappa/2$.*

3.2. **Some identities in the curvinormal basis.** Recall that if u is a vector, $u = u_i g^i = \hat{u}_j e^j$, then $(\nabla u)_{mn} = \partial_{x_m}\hat{u}_n$, and we write

$$\mathrm{grad} u := (\nabla u)_{mn} e^m \otimes e^n = u_{i\|j} g^i \otimes g^j = \partial_j u \otimes g^j. \tag{3.11}$$

Let q_R be the curvilinear coordinate associated to g^R in the sense that $g^R = \partial_R x = \frac{1}{h_R}\partial_{q_R} x$, with $h_R := \|\partial_{q_R} x\|$, and where x stands for the position vector of a point. Otherwise said, q_R is the curvilinear abcissa of the curve with tangent vector τ^R. Indeed,

$$\partial_{q_R} u = \partial_{x_i} u \frac{\partial x_i}{\partial q_R} = h_R g_i^R \partial_{x_i} u = h_R \partial_R u. \tag{3.12}$$

Remark that by Lemma 2.3 the ∂_R derivatives do not commute, contrarily to ∂_{q_R}, because of the factors h_R. This object of anholonomicity (cf. [17, 18]) reflects itself in the non-symmetry of the Christoffel symbols. As an example, consider the spherical base system, where $h_N = h_r = 1, h_A = h_\phi = \frac{1}{r}, h_B = h_\theta = \frac{1}{r\tan\phi}$, and $q_A = \phi$ (polar angle), $q_B = \theta$ (azimuthal angle). One has

$$(\partial_A\partial_r - \partial_r\partial_A) = \frac{1}{r^2}\partial_\phi,\ (\partial_B\partial_r - \partial_r\partial_B) = \frac{1}{r^2\sin\phi}\partial_\theta,\ (\partial_B\partial_A - \partial_A\partial_B) = \frac{1}{r^2\sin\phi\tan\phi}\partial_\theta \tag{3.13}$$

Partial derivative in curvilinear coordinates might be defined in a weak sense by means of surface integrals and (3.12) as follows:

$$\int_{\omega^A\times\omega^B} \partial_R u \cdot \varphi J dq_A dq_B := -\int_{\omega^A\times\omega^B} u \cdot h_R \partial_R \left(\frac{1}{h_R}\varphi J\right) dq_A dq_B, \tag{3.14}$$

with J the surface Jacobian and φ, a test function with compact support. Here ω^R stands for a subset of the domain of q_R.

Setting

$$e_{ij} = e_{ij}(u) := \frac{1}{2}(u_{i\|j} + u_{j\|i}), \tag{3.15}$$

one has by (3.2) that

$$e_{ij} = \frac{1}{2}(\partial_i u \cdot g_j + \partial_j u \cdot g_i), \tag{3.16}$$

and in particular,

$$e_{NN} = u_{N\|N} = (\partial_N u)\cdot N. \tag{3.17}$$

In general, for the covariant components of a second-rank tensor A it holds [8]

$$A_{ij\|k} = \partial_k A_{ij} - \Gamma^l_{ik} A_{lj} - \Gamma^l_{jk} A_{il}. \tag{3.18}$$

Let $A_{ij} = u_{i\|j}$. Then by (3.18) one has $u_{i\|jk} := (u_{i\|j})_{\|k}$ and hence

$$\begin{aligned} u_{i\|jk} g^i &= \partial_k u_{i\|j} g^i - (\Gamma^l_{ik} u_{l\|j} + \Gamma^l_{jk} u_{i\|l}) g^i \\ &= \partial_k \left(u_{i\|j} g^i\right) + u_{i\|j}\Gamma^i_{lk} g^l - (\Gamma^l_{ik} u_{l\|j} + \Gamma^l_{jk} u_{i\|l}) g^i \\ &= \partial_k \left(u_{i\|j} g^i\right) - \Gamma^l_{jk} u_{i\|l} g^i, \end{aligned}$$

where (3.3) and a change of dumb indices have been used. Therefore,

$$\partial_k(\partial_j u) = \left(u_{i\|jk} + \Gamma^l_{jk} u_{i\|l}\right) g^i. \tag{3.19}$$

In particular, it follows by (3.2) that

$$\partial_k(\partial_j u)\cdot N = u_{N\|jk} + \big(\Gamma^l_{jk}\partial_l u\big)\cdot N. \tag{3.20}$$

Moreover, one has

$$\begin{aligned}\partial_k(\partial_j u)\cdot\tau^R = \partial_k(\partial_j u\cdot\tau^R) - \partial_j u\cdot\partial_k\tau^R &= \partial_k(\partial_j u_R) - \partial_k(u\cdot\partial_j\tau^R) - \partial_j u\cdot\partial_k\tau^R\\ &= u_{R\|jk} + \big(\Gamma^l_{jk}\partial_l u\big)\cdot\tau^R. \qquad (3.21)\end{aligned}$$

Remark also that

$$u_{i\|jk} - u_{i\|kj} = (\partial_k\partial_j - \partial_j\partial_k)u\cdot g^i - T^l_{jk}u_{i\|l}, \tag{3.22}$$

where we recall that T stands for the anholonomicity torsion. Note that in spherical coordinates and by (3.6) and (3.13) reads (with a slight abuse of notations, since one writes q_R instead of R as covariant differentiation indice)

$$u_{i\|r\phi} - u_{i\|\phi r} = u_{i\|r\theta} - u_{i\|\theta r} = u_{i\|\theta\phi} - u_{i\|\phi\theta} = 0, \tag{3.23}$$

that is, the second covariant derivatives commute in spherical coordinates/components. In particular, one has $\epsilon_{ljk}u_{i\|jk} = 0$ in spherical coordinates/components.

Some identities are most easily derived for the contravariant components, and only expressed in covariant components in a second step. Let u^i be the contravariant coordinate of the vector $u = u^i g_i$. Then it is well known that the covariant derivative of vector u and tensor A are expressed as (cf. [8]), $u^i_{\|j} := \partial_j u^i + \Gamma^i_{pj}u^p$ and $(A^i_j)_{\|k} := \partial_k A^i_j + \Gamma^i_{pk}A^p_j - \Gamma^p_{jk}A^i_p$. Moreover, $u^i_{\|jk} := (u^i_{\|j})_{\|k}$, and the Ricci identity for contravariant components u^i then reads (with sum on q and p)

$$u^i_{\|jk} - u^i_{\|kj} = \mathrm{Riem}^i_{qjk}u^q + T^p_{jk}u^i_{\|p},$$

which is an alternative formula to (3.22) which shows only the value of u and of its derivative in the RHS. Therefore,

$$u^Q_{\|NR} - u^Q_{\|RN} = \mathrm{Riem}^Q_{qNR}u^q + T^p_{NR}u^Q_{\|p},$$

which reads in the curvinormal basis (with sum on q but not on R),

$$\begin{aligned}u^Q_{\|NR} - u^Q_{\|RN} &= \partial_N\Gamma^Q_{qR}u^q + T^P_{NR}u^Q_{\|P} = \partial_N\Gamma^Q_{qR}u^q + \kappa^R u^Q_{\|R} + \xi u^Q_{\|R^*}\\ &= \left(\partial_N\Gamma^Q_{qR} + \kappa^R\Gamma^Q_{qR} + \xi\Gamma^Q_{qR^*}\right)u^q + \kappa^R\partial_R u^Q + \xi\partial_{R^*}u^Q\\ &= \ell^Q_{NR;q}u^q + \kappa^R\partial_R u^Q + \xi\partial_{R^*}u^Q, \qquad (3.24)\end{aligned}$$

where ℓ is the expression inside the parenthesis of the RHS. Setting apart the linear parts in u^q, and in $\partial_R u^q$, we have expressed (3.24) as

$$\delta u^Q_{\|NR} := u^Q_{\|NR} - u^Q_{\|RN} = \ell^Q_{NR;q}u^q + \tilde\ell_{NR}\partial_R u^Q + \tilde\ell_{NR^*}\partial_{R^*}u^Q, \tag{3.25}$$

where $\tilde\ell$ depend on κ and ξ. It is obvious that the covariant differentiation of the curvinormal metric $g_{ij\|k}$ vanishes, since the metric is the identity tensor. Thus, lowering indices and covariant differentiation mutually commute [8] and the counterpart of (3.25) for the covariant components reads

$$\delta u_{m\|jR} := u_{m\|jR} - u_{m\|Rj} = \delta_{Qm}\ell^Q_{NR;q}u^q + \tilde\ell_{NR}\partial_R u_m + \tilde\ell_{NR^*}\partial_{R^*}u_m. \tag{3.26}$$

where $\ell^Q_{mNR;q} := \delta_{Qm}\ell^Q_{NR;q}$. Now, it is easily deduced from (3.15) and (3.25) that

$$e_{QN\|R} + e_{RN\|Q} - e_{QR\|N} = u_{N\|QR} + \frac{1}{2}(\delta u_{Q\|NR} + \delta u_{R\|NQ}),$$

which by (3.3) and (3.20) yields

$$\begin{aligned}e_{QN\|R} + e_{RN\|Q} - e_{QR\|N} &= u_{N\|QR} + \frac{1}{2}(\delta u_{Q\|NR} + \delta u_{R\|NQ})\\ &= \big(\partial_R\partial_Q u - \Gamma^l_{QR}\partial_l u\big)\cdot N + \frac{1}{2}(\delta u_{Q\|NR} + \delta u_{R\|NQ}).\end{aligned}$$

Hence, by (3.17),

$$\begin{aligned} e_{QN\|R} + e_{RN\|Q} - e_{QR\|N} + \Gamma^N_{QR} e_{NN} = & \quad \left(\partial_R \partial_Q u - \Gamma^P_{QR} \partial_P u\right) \cdot N \\ & + \frac{1}{2}(\delta u_{Q\|NR} + \delta u_{R\|NQ}). \end{aligned} \tag{3.27}$$

3.3. **Restatement of Ciarlet-Mardare results in** [5]**.** Basically, the differential geometry on the boundary by Ph. Ciarlet and C. Mardare [3, 5] is defined by means of a holonomic non-orthogonal frame, whereas our choice of basis vectors on the boundary, suitably extended in the domain, are orthonormal, but non-holonomic, thereby inducing a non-vanishing anholonomicity (non-symmetry of the Christoffels symbols).

Let Γ_0 be a connected subset of $\partial\Omega$. Let $u \in \mathscr{C}^2(\bar{\Omega})$ and let u_{Γ_0} denote the boundary trace of u on Γ_0. Set

$$\begin{aligned} e = e(u) := \operatorname{grad}^S u = (\nabla^S \hat{u})_{mn} e^m \otimes e^n &= \frac{1}{2}(u_{i\|j} + u_{j\|i}) g^i \otimes g^j \\ &= \frac{1}{2}(\partial_i u \cdot g^j + \partial_j u \cdot g^i) g^i \otimes g^j, \end{aligned} \tag{3.28}$$

and let e_{Γ_0} be the boundary trace of e on Γ_0. Let us define the operator $\gamma : \mathscr{C}^2(\bar{\Gamma}_0, \mathbb{R}^3) \to \mathscr{C}^1(\bar{\Gamma}_0, \mathbb{R}^3)$,

$$\gamma(u_{\Gamma_0}) := \gamma_{QR}(u_{\Gamma_0}) g^Q \otimes g^R \in \mathscr{C}^1(\bar{\Gamma}_0, \mathbb{R}^3), \tag{3.29}$$

called the *linearized change of metric* induced by u_{Γ_0} [5], whose components read

$$\gamma_{QR}(u_{\Gamma_0}) := \frac{1}{2}(\partial_Q u_{\Gamma_0} \cdot g^R + \partial_R u_{\Gamma_0} \cdot g^Q). \tag{3.30}$$

Furthermore, we introduce the operator $\gamma^\sharp : \mathscr{C}^1(\bar{\Omega}) \to \mathscr{C}^1(\bar{\Gamma}_0, \mathbb{R}^3)$

$$\gamma^\sharp(e) := \gamma^\sharp_{QR}(e) g^Q \otimes g^R \in \mathscr{C}^1(\bar{\Gamma}_0, \mathbb{R}^3), \tag{3.31}$$

where

$$\gamma^\sharp_{QR}(e) := (e_{\Gamma_0})_{QR}. \tag{3.32}$$

It is immediate from (3.28) and our curvinormal frame approach that

$$\gamma^\sharp_{QR}(e) = \gamma_{QR}(u_{\Gamma_0}). \tag{3.33}$$

Definition 3.2. *Let the linearized change of curvature induced by u_{Γ_0} be given by*

$$\rho(u_{\Gamma_0}) := \rho_{QR}(u_{\Gamma_0}) g^Q \otimes g^R \in \mathscr{C}(\bar{\Gamma}_0, \mathbb{R}^3) \tag{3.34}$$

with the components given by

$$\rho_{QR}(u_{\Gamma_0}) := \left(\partial_R \partial_Q u_{\Gamma_0} - \Gamma^P_{QR} \partial_P u_{\Gamma_0}\right) \cdot N. \tag{3.35}$$

Let

$$\rho^\sharp(e) := \rho^\sharp_{QR}(e) g^Q \otimes g^R \in \mathscr{C}(\bar{\Gamma}_0, \mathbb{R}^3), \tag{3.36}$$

with the components given by

$$\rho^\sharp_{QR}(e) := \left(e_{QN\|R} + e_{RN\|Q} - e_{QR\|N} + \Gamma^N_{QR} e_{NN}\right)_{\Gamma_0}. \tag{3.37}$$

The main preliminary results of [5] are restated in the curvinormal basis as follows:

Theorem 3.3 (Ciarlet-Mardare [5])**.** *One has*

$$\rho^\sharp(e) = \rho(u_{\Gamma_0}) + \delta\rho(u_{\Gamma_0}) \quad on \quad \Gamma_0, \tag{3.38}$$

where

$$(\delta\rho)_{QR}(u) := \frac{1}{2}(\delta u_{Q\|NR} + \delta u_{R\|NQ}). \tag{3.39}$$

Moreover, there exist positive constants C_1, C_2, C_3 and C_4 such that

$$\|\gamma^\sharp(e)\|_{H^{-1}(\Gamma_0)} + \|\rho^\sharp(e)\|_{H^{-2}(\Gamma_0)} \leq C_1 \inf_{r \in \mathcal{R}(\Omega)} \|(u + r)_{\Gamma_0}\|_{L^2(\Gamma_0)} \leq C_2 \|e\|_{L^2(\Omega)}, \tag{3.40}$$

with $\mathcal{R}(\Omega)$ the set of rigid displacements in Ω, and

$$\begin{aligned}\inf_{r\in\mathcal{R}(\Gamma_0)}\|(u+r)_{\Gamma_0}\|_{L^2(\Gamma_0)} &\leq C_3\left(\|\gamma(u_{\Gamma_0})\|_{H^{-1}(\Gamma_0)}+\|\rho(u_{\Gamma_0})\|_{H^{-2}(\Gamma_0)}\right) &(3.41)\\ &\leq C_4\left(\|\gamma(u_{\Gamma_0})\|_{H^{-1}(\Gamma_0)}+\|(\rho+\delta\rho)(u_{\Gamma_0})\|_{H^{-2}(\Gamma_0)}\right) &(3.42)\\ &\leq C_4\left(\|\gamma^\sharp(e)\|_{H^{-1}(\Gamma_0)}+\|\rho^\sharp(e)\|_{H^{-2}(\Gamma_0)}\right), &(3.43)\end{aligned}$$

with $\mathcal{R}(\Gamma_0)$ the set of rigid displacement in Γ_0.

Note that (3.43) stems from (3.42) by Eqs. (3.33) and (3.38). Note also that in Ciarlet-Mardare original formulation of this result, $\delta\rho = 0$ because their connection is symmetric (i.e., their frame is holonomic). The proof of (3.38) is basically Eq. (3.27), here proved in the curvinormal basis. By inspecting the original proof in [5], Eq. (3.40) is also easily demonstrated, since by formulae (3.25) and (3.14) one has

$$|\int_{\Gamma_0}(\delta\rho)_{QR}(u)\cdot\varphi dS(x)|\leq|\int_{\omega^0}u\cdot\Psi dq^Adq^B\|\leq C_1\|u_{\Gamma_0}\|_{L^2(\Gamma_0)},$$

for some positive constant C_1, some vector Ψ with compact support independent of u, and with ω^0 the domain of (q^A,q^B) associated to Γ_0. Lastly, to prove (3.41), scrutating again the original proof in [5], it suffices to bound the terms $\partial_R\partial_Q(u_P)_{\Gamma_0}$ in $H^{-2}(\omega^0)$ in terms of $\|\frac{1}{2}(\partial_R(u_{\Gamma_0})_Q+\partial_Q(u_{\Gamma_0})_R)\|_{H^{-1}(\omega^0)}$ and $\|u_{\Gamma_0}\|_{H^{-1}(\omega^0)}$, then use an argument of Nečas [2], noting that the extra terms appearing due to the nonsymmetric property of the connexion are also bounded by $\|\frac{1}{2}(\partial_R(u_{\Gamma_0})_Q+\partial_Q(u_{\Gamma_0})_R)\|_{H^{-1}(\omega^0)}$ and by $\|u_{\Gamma_0}\|_{H^{-1}(\omega^0)}$, from formulae (3.21) and (3.25). Note that the bound (3.42) is obtained by writing $\|\rho(u_{\Gamma_0})\|_{H^{-2}(\Gamma_0)}\leq\|(\rho+\delta\rho)(u_{\Gamma_0})\|_{H^{-2}(\Gamma_0)}+\|\delta\rho(u_{\Gamma_0})\|_{H^{-2}(\Gamma_0)}$ and using the fact that by (3.39) and (3.24) one bounds $\|\delta\rho(u_{\Gamma_0})\|_{H^{-2}(\Gamma_0)}$ by means of $\|\frac{1}{2}(\partial_R(u_{\Gamma_0})_Q+\partial_Q(u_{\Gamma_0})_R)\|_{H^{-1}(\omega^0)}$ and $\|u_{\Gamma_0}\|_{H^{-1}(\omega^0)}$.

The main result in [5] is now stated in the curvinormal basis. Its proof basically follows from (3.38)-(3.41) and standard extension operators.

Theorem 3.4 (Ciarlet-Mardare Main result [5])**.** *Let $u\in H^1(\Omega)$ and let $e=e_{ij}(u)g^i\otimes g^j$ with*

$$e_{ij}(u)=\frac{1}{2}(\partial_i u\cdot g^j+\partial_j u\cdot g^i). \tag{3.44}$$

Then the following conditions satisfy (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii):

(i) $u_{\Gamma_0}=0$
(ii) $\bar\gamma^\sharp(e)=\bar\rho^\sharp(e)=0$
(iii) $u_{\Gamma_0}\in\mathcal{R}(\Gamma_0)$,

where $\bar\gamma^\sharp$ and $\bar\rho^\sharp$ are suitable extensions of $\gamma^\sharp$ and $\rho^\sharp$.

4. Function spaces for the incompatibility operator

4.1. Definitions and basic properties.

Let Γ_0 be a connected subset of $\partial\Omega$. Define

$$\begin{aligned}\mathcal{H}^2_{\rm comp}(\Omega) &:= \{E\in H^2(\Omega,\mathbb{S}^3):\ \text{inc } E=0 \text{ in } \Omega\}\\ \mathcal{H}^2_{\Gamma_0;\rm comp}(\Omega) &:= \{E\in\mathcal{H}^2_{\rm comp}(\Omega):\ E=\text{Curl}^{\rm t}\ E\times N=0 \text{ on } \Gamma_0\}\\ \mathcal{H}_{\rm inc}(\Omega) &:= \{E\in L^2(\Omega,\mathbb{S}^3):\ \text{inc } E\in L^2(\Omega,\mathbb{S}^3)\}\\ \mathcal{H}_{\rm comp}(\Omega) &:= \{E\in L^2(\Omega,\mathbb{S}^3):\ \text{inc } E=0 \text{ in } \Omega\}\\ \mathcal{H}_{\Gamma_0;\rm comp}(\Omega) &:= \{E\in\mathcal{H}_{\rm inc}(\Omega):\ \text{inc } E=0 \text{ in } \Omega,\ E=\text{Curl}^{\rm t}\ E\times N=0 \text{ on } \Gamma_0\}.\end{aligned}$$

The spaces $\mathcal{H}(\Omega)$, $\mathcal{H}_0(\Omega)$ and the above affine spaces are naturally endowed with the Hilbertian structure of $H^2(\Omega,\mathbb{S}^3)$. Note that in order to define $\mathcal{H}_{\rm inc}(\Omega)$ we should precise that inc $E\in H^{-2}(\Omega,\mathbb{S}^3)$ and that the boundary traces in the space $\mathcal{H}_{\rm comp}(\Omega)$ are defined in a weak sense, whose exact meaning is the object of Corollary 4.5 (see Eqs. (4.18) and (4.19)).

Lemma 4.1 (Amstutz-Van Goethem [1])**.** *For all $E\in H^2(\Omega,\mathbb{S}^3)$ it holds an open neighborhood W of $\partial\Omega$:*

$$\text{Curl}^{\rm t}\ E\times N=-\left(\partial_N E\times N\right)^t\times N+\left(\sum_R\tau^R\times\partial_R E\right)^t\times N \qquad \text{on } \partial\Omega.$$

In particular, if $(\partial_N E \times N)^t \times N = 0 = E$ *on* $\partial\Omega$*, then*

$$\mathrm{Curl}^{\mathrm{t}}\ E \times N = 0 \qquad \text{on } \partial\Omega.$$

Lemma 4.2. *Let* $E \in H^2(\Omega, \mathbb{S}^3)$*. If* $(\mathrm{Curl}^{\mathrm{t}}\ E) \times N = 0$ *on on* Γ_0*, then (i)* $\partial_R E_{RN} = \partial_N E_{RR}$*, (ii)* $\partial_R E_{R^*N} = \partial_N E_{RR^*}$*, and (iii)* $\partial_R E_{R^*R^*} - \partial_R E_{RR^*} = 0$*, on on* Γ_0 *and for* $R = A$ *or* $R = B$*. In particular, if* $E = 0$ *on on* Γ_0 *then* $\partial_N E_{RR} = \partial_N E_{RR^*} = 0$ *on on* Γ_0*.*

Proof. Let us compute $(\mathrm{Curl}^{\mathrm{t}}\ E) \times N$ in the local basis (τ^A, τ^B, N). From

$$(\mathrm{Curl}^{\mathrm{t}}\ E) \times N)_{im} = \epsilon_{mjp}\epsilon_{ikl}\partial_k E_{jl} N_p,$$

one has $N_p = \delta_{pN}$ and statement (i) follows from (with $R = A, B$)

$$(\mathrm{Curl}^{\mathrm{t}}\ E) \times N)_{R^*R^*} = \epsilon_{R^*jN}\epsilon_{R^*kl}\partial_k E_{jl} = (\delta_{Rk}\delta_{Nl} - \delta_{Rl}\delta_{Nk})\partial_k E_{Rl} = \partial_R E_{RN} - \partial_N E_{RR}.$$

Statement (ii) follows from

$$(\mathrm{Curl}^{\mathrm{t}}\ E) \times N)_{RR^*} = \epsilon_{R^*jN}\epsilon_{Rkl}\partial_k E_{jl} = \epsilon_{R^*RN}\epsilon_{Rkl}\partial_k E_{jl} = \partial_N E_{RR^*} - \partial_{R^*} E_{RN},$$

and (iii) by

$$(\mathrm{Curl}^{\mathrm{t}}\ E) \times N)_{NR} = \epsilon_{RjN}\epsilon_{Nkl}\partial_k E_{jl} = \epsilon_{Nkl}\partial_k E_{R^*l} = \partial_R E_{R^*R^*} - \partial_{R^*} E_{R^*R},$$

with $R = A, B$, proving the result, since $\partial_R E = E = 0$ on on Γ_0 if $E = 0$ on on Γ_0. □

The central contribution of the present work is the following Theorem, whose proof follows from the preceeding discussion.

Theorem 4.3. *Let* $e \in \mathscr{C}^1(\bar{\Omega}) \cap \mathcal{H}^2_{\Gamma_0;\mathrm{comp}}(\Omega)$*. The following conditions are equivalent:*

(1) $e = \mathrm{Curl}^{\mathrm{t}}\ e \times N = 0$ *on* Γ_0
(2) $\gamma^\sharp(e) = \rho^\sharp(e) = 0$ *on* Γ_0*.*

Proof. This is a direct consequence of Eqs. (3.33) and (3.37), Eq. (3.18), and items (i) and (ii) of Lemma 4.2. □

4.2. **Green formula and weak trace operators.** Denote $A^S = (A + A^T)/2$ the symmetric part of a tensor A.

Theorem 4.4 (Green formula for the incompatibility[1])**.** *Suppose that* $E \in \mathcal{C}^2(\bar{\Omega}, \mathbb{S}^3)$ *and* $\eta \in H^2(\Omega, \mathbb{S}^3)$*. Then*

$$\begin{aligned} \int_\Omega E \cdot \mathrm{inc}\ \eta dx &= \int_\Omega \mathrm{inc}\ E \cdot \eta dx \\ &+ \int_{\partial\Omega} \mathcal{T}_1(E) \cdot \eta\ dS(x) + \int_{\partial\Omega} \mathcal{T}_0(E) \cdot \partial_N \eta dS(x), \end{aligned} \tag{4.1}$$

with the trace operators defined as

$$\mathcal{T}_0(E) := (E \times N)^t \times N, \tag{4.2}$$

$$\mathcal{T}_1(E) := \left(\mathrm{Curl}^{\mathrm{t}}\ (E \times N)\right)^S + ((\partial_N + \kappa)E \times N)^t \times N + \left(\mathrm{Curl}^{\mathrm{t}}\ E \times N\right)^S, \tag{4.3}$$

where we recall that κ *stands for twice the mean curvature (cf. Section 2.3).*

It is crucial to note that, by (4.2), only the tangential components of $\partial_N \eta$ are to be considered in the right-hand side of (4.1).

As we have seen, the two Dirichlet boundary conditions imposed in the space $\mathcal{H}^2_{\Gamma_0;\mathrm{comp}}(\Omega)$ read $\mathscr{D}_1(E) = E_{\Gamma_0}$ (i.e., the trace of E on Γ_0) and $\mathscr{D}_2(E) = (\mathrm{Curl}^{\mathrm{t}}\ E \times N)_{\Gamma_0}$ (i.e., the trace of $\mathrm{Curl}^{\mathrm{t}}\ E \times N$ on Γ_0, where N is a suitable extension of the normal). In particular, let

$$\mathscr{E}_{\mathrm{comp}}(\Omega) := \{\nabla^S u : u \in \mathscr{C}^2(\bar{\Omega}, \mathbb{R}^3)\} \subset \mathscr{C}^1(\bar{\Omega}, \mathbb{S}^3).$$

One has

$$\mathscr{D}_1 : \mathscr{C}^1(\bar{\Omega}, \mathbb{S}^3) \to \mathscr{C}^1(\bar{\Gamma}_0) \tag{4.4}$$

$$E \in \mathscr{E}_{\mathrm{comp}}(\Omega) \mapsto \mathscr{D}_1(E) = E_{\Gamma_0} \in \mathscr{C}^1(\bar{\Gamma}_0), \tag{4.5}$$

and

$$\mathscr{D}_2 : \mathscr{C}^1(\bar{\Omega}, \mathbb{S}^3) \to \mathscr{C}(\bar{\Gamma}_0) \tag{4.6}$$

$$E \in \mathscr{E}_{\rm comp}(\Omega) \mapsto \mathscr{D}_2(E) = (\,{\rm Curl}^{\rm t}\ E \times N)_{\Gamma_0} \in \mathscr{C}(\bar{\Gamma}_0). \tag{4.7}$$

In the following result, trace operators will be introduced in a weak form. While $\overline{\mathcal{T}}_0$ and $\overline{\mathcal{T}}_1$ are extensions of (4.2) and (4.3), respectively, the operators $\bar{\mathscr{D}}_2$ and $\bar{\mathscr{D}}_1$ stand for the traces of ${\rm Curl}^{\rm t}\ E \times N$ and E on $\partial\Omega$, respectively.

Corollary 4.5 (Weak trace operators)**.** *Let $\Phi \in H^2(\Omega, \mathbb{S}^3)$, with $\gamma_0(\Phi) \in H^{3/2}(\partial\Omega, \mathbb{S}^3)$ the boundary trace of Φ. Let $\gamma_i(\Phi) \in H^{1/2}(\partial\Omega, \mathbb{S}^3)$ $(i = 1, 2)$ the boundary traces of $\nabla\Phi \cdot N$ and ${\rm Curl}^{\rm t}\ \Phi \times N$ for $i = 1$ or $i = 2$, respectively. Let $E \in \mathcal{H}_{\rm inc}(\Omega)$.*

Then there exists[2] $\overline{\mathcal{T}}_0(E) \in H^{-1/2}(\partial\Omega) := \big(H^{1/2}(\partial\Omega, \mathbb{S}^3)\big)'$ *such that*

$$\langle \overline{\mathcal{T}}_0(E), \gamma_1(\Phi)\rangle := \langle E, \text{ inc } \Phi\rangle - \langle \text{ inc } E, \Phi\rangle, \tag{4.8}$$

for all Φ such that $\gamma_0(\Phi) = 0$. Moreover, there exists $\overline{\mathcal{T}}_1(E) \in H^{-3/2}(\partial\Omega) := \big(H^{3/2}(\partial\Omega, \mathbb{S}^3)\big)'$ such that

$$\langle \overline{\mathcal{T}}_1(E), \gamma_0(\Phi)\rangle := \langle E, \text{ inc } \Phi\rangle - \langle \text{ inc } E, \Phi\rangle, \tag{4.9}$$

for all Φ such that $\gamma_1(\Phi) = 0$.

Furthermore, there exists $\bar{\mathscr{D}}_2(E) \in H^{-3/2}(\partial\Omega)$ such that

$$\langle \bar{\mathscr{D}}_2(E), \gamma_0(\Phi)\rangle := \langle E, \text{ inc } \Phi\rangle - \langle \text{ inc } E, \Phi\rangle, \tag{4.10}$$

for all Φ such that $\gamma_2(\Phi) = 0$. Moreover, there exists $\bar{\mathscr{D}}_1(E) \in H^{-1/2}(\partial\Omega)$ such that

$$\langle \bar{\mathscr{D}}_1(E), \gamma_2(\Phi)\rangle := \langle \text{ inc } E, \Phi\rangle - \langle E, \text{ inc } \Phi\rangle, \tag{4.11}$$

for all Φ such that $\gamma_0(\Phi) = 0$, respectively.

Here symbol $\langle\cdot\rangle$ stands for the of duality product in appropriate spaces.

Proof. By the Green formula (4.1), let us define the linear functional on $H^{1/2}(\partial\Omega)$ by

$$\langle \overline{\mathcal{T}}_0(E), \gamma_1\rangle := \langle E, \text{ inc } \Phi\rangle - \langle \text{ inc } E, \Phi\rangle,$$

where $\Phi \in H^2(\Omega, \mathbb{S}^3)$ satisfies $\gamma_0(\Phi) = 0$ and $\gamma_1(\Phi) = \gamma_1$ for a given $\gamma_1 \in H^{3/2}(\partial\Omega)$; define also the linear functional on $H^{3/2}(\partial\Omega)$ by

$$\langle \overline{\mathcal{T}}_1(E), \gamma_0\rangle := \langle E, \text{ inc } \Phi\rangle - \langle \text{ inc } E, \Phi\rangle,$$

where $\Phi \in H^2(\Omega, \mathbb{S}^3)$ satisfies $\gamma_1(\Phi) = 0$ and $\gamma_0(\Phi) = \gamma_0$ for a given $\gamma_0 \in H^{1/2}(\partial\Omega)$. First observe that $\overline{\mathcal{T}}_i(E)$ $(i = 0, 1)$ does not depend on the chosen extension. If Φ_1, Φ_2 are two such extensions, then their difference has zero trace and

$$0 = \langle E, \text{ inc } (\Phi_1 - \Phi_2)\rangle - \langle \text{ inc } E, \Phi_1 - \Phi_2\rangle,$$

by (4.1), since $\gamma_0(\Phi_1 - \Phi_2) = \gamma_1(\Phi_1 - \Phi_2) = 0$. It has been proved in [1] that a lifting operator $\mathcal{L}_{\partial\Omega} : H^{1/2}(\partial\Omega, \mathbb{S}^3) \times H^{3/2}(\partial\Omega, \mathbb{S}^3) \to H^2(\Omega, \mathbb{S}^3)$ exists and can be chosen so that by its linearity and continuity (note that in Lemma 3.11. of [1] such a lifting can be taken solenoidal on the boundary), it holds $(i = 0, 1)$

$$\begin{aligned} |\langle \overline{\mathcal{T}}_{i^*}(E), \gamma_i\rangle| &\leq C\,(\|E\|_{L^2} + \|\text{ inc } E\|_{L^2})\,\|\mathcal{L}_{\partial\Omega}(\gamma_i)\|_{H^2(\Omega)} \\ &\leq C\,(\|E\|_{L^2} + \|\text{ inc } E\|_{L^2})\,\|\gamma_i\|_{H^{3/2-i}(\partial\Omega)}, \end{aligned}$$

achieving the proof of the first two statements.

As for the last one, take $E \in \mathcal{C}^2(\bar{\Omega}, \mathbb{S}^3)$ and $\eta \in H^2(\Omega, \mathbb{S}^3)$ such that $\gamma_2(\eta) = 0$ and compute by a series of integrations by parts,

$$\begin{aligned} \int_\Omega E \cdot \text{ inc } \eta dx &= \int_\Omega \text{Curl } E \cdot \text{Curl}^{\rm t}\ \eta dx = \int_\Omega \text{Curl}^{\rm t}\ E \cdot \text{Curl } \eta dx \\ &= \int_\Omega \text{inc } E \cdot \eta dx + \int_{\partial\Omega} \text{Curl}^{\rm t}\ E \times N \cdot \eta dS(x). \end{aligned} \tag{4.12}$$

[2] There is no need to write $H_0^{1/2}(\partial\Omega, \mathbb{S}^3)$ since the boundary is a closed surface and this space is defined by density of smooth functions with compact support.

By (4.7), one has

$$\langle \mathscr{D}_2(E), \gamma_0\rangle = \langle E, \text{inc } \Phi\rangle - \langle \text{ inc } E, \Phi\rangle, \tag{4.13}$$

where $\Phi \in H^2(\Omega, \mathbb{S}^3)$ satisfies $\gamma_0(\Phi) = \gamma_0$ and $\gamma_2(\Phi) = 0$. Now, let $E \in \mathcal{H}_{\text{inc}}(\Omega)$ and define

$$\langle \bar{\mathscr{D}}_2(E), \gamma_0\rangle := \langle E, \text{inc } \Phi\rangle - \langle \text{ inc } E, \Phi\rangle \tag{4.14}$$

where $\Phi \in H^2(\Omega, \mathbb{S}^3)$ satisfies $\gamma_0(\Phi) = \gamma_0$ and $\gamma_2(\Phi) = 0$. By the above lifting operator $\mathcal{L}_{\partial\Omega}$ and provided Lemma 4.1 (which states that given the curl transpose and the value of E on the boundary yields the tangential components of $\partial_N E$), one obtains

$$|\langle \bar{\mathscr{D}}_2(E), \gamma_0\rangle| \le C\,(\|E\|_{L^2} + \|\text{ inc } E\|_{L^2})\, \|\gamma_0\|_{H^{3/2}(\partial\Omega)}, \tag{4.15}$$

whence linearity and continuity of $\bar{\mathscr{D}}_2$ in $H^{-3/2}(\partial\Omega)$.

Inverting the roles of E and η in (4.12), and defining

$$\langle \bar{\mathscr{D}}_1(E), \gamma_2\rangle := \langle \text{ inc } E, \Phi\rangle - \langle E, \text{ inc } \Phi\rangle, \tag{4.16}$$

where $\Phi \in H^2(\Omega, \mathbb{S}^3)$ satisfies $\gamma_0(\Phi) = 0$ and $\gamma_2(\Phi) = \gamma_2$, also yields linearity and continuity of $\bar{\mathscr{D}}_1$ in $H^{-1/2}(\partial\Omega)$, achieving the proof. ☐

Obviously (4.17) holds for any $\gamma_0 \in \mathscr{C}_c^\infty(\Gamma_0)$ and hence

$$|\langle \bar{\mathscr{D}}_2(E), \gamma_0\rangle| \le C\,(\|E\|_{L^2} + \|\text{ inc } E\|_{L^2})\, \|\gamma_0\|_{H^{3/2}(\Gamma_0)}, \tag{4.17}$$

for any $\gamma_0 \in H_0^{3/2}(\Gamma_0)$. A similar reasoning can be made for $\mathscr{D}_1$ and hence (4.5) and (4.7) can be extended as follows:

$$\bar{\mathscr{D}}_{1,0}\mathcal{H}_{\text{inc}}(\Omega) \to H^{-1/2}(\Gamma_0) \tag{4.18}$$

$$\bar{\mathscr{D}}_{2,0} : \mathcal{H}_{\text{inc}}(\Omega) \to H^{-3/2}(\Gamma_0). \tag{4.19}$$

4.3. **Saint-Venant conditions and Beltrami decomposition.** The following result is given for the sake of generality in $L^p(\Omega)$ with $1 < p < \infty$ but should here be considered for $p = 2$.

Theorem 4.6 (Saint-Venant compatibility conditions in L^p [14])**.** *Let $\Omega \subseteq \mathbb{R}^3$ be a simply-connected domain, let $1 < p < +\infty$, and let $E \in L^p(\Omega, \mathbb{S}^3)$ be a symmetric tensor. Then*

$$\text{inc } E = 0 \text{ in } W^{-2,p}(\Omega, \mathbb{S}^3) \iff E = \nabla^S u$$

for some $u \in W^{1,p}(\Omega, \mathbb{R}^3)$. Moreover, u is unique up to rigid displacements.

The following result is again given for the sake of generality in $L^p(\Omega)$ with $1 < p < \infty$ but should here be considered for $p = 2$.

Theorem 4.7 (Beltrami decomposition in L^p [14])**.** *Assume that Ω is simply-connected. Let $p \in (1, +\infty)$ be a real number and let $E \in L^p(\Omega, \mathbb{S}^3)$. Then, For any $u_0 \in W^{1/p,p}(\partial\Omega)$, there exists a unique $u \in W^{1,p}(\Omega, \mathbb{R}^3)$ with $u = u_0$ on $\Gamma_0 \subset \partial\Omega$ and a unique $F \in L^p(\Omega, \mathbb{S}^3)$ with* Curl $F \in L^p(\Omega, \mathbb{R}^{3\times3})$, inc $F \in L^p(\Omega, \mathbb{S}^3)$, div $F = 0$ *and $FN = 0$ on $\partial\Omega$ such that*

$$E = \nabla^S u + \text{ inc } F. \tag{4.20}$$

We call $\nabla^S u$ the compatible part and inc F *the (solenoidal) incompatible part of the Beltrami decomposition.*

Observe that if inc $E = 0$ in Ω and $u_0 = 0$ on Γ_0 then u is uniquely determined such that

$$E = \nabla^S u, \tag{4.21}$$

since $F = 0$ is the unique solution of the decomposition. Indeed, u is the unique solution of

$$-\text{ div } \left(\nabla^S u\right) = -\text{ div } E \quad \text{in} \quad \Omega,$$

with as boundary conditions, $u = 0$ on Γ_0 and $(\nabla^S u)N = EN$ on $\partial\Omega \setminus \Gamma_0$. Now, if there were two solutions F_1 and F_2 satisfying (4.20), then inc $(F_1 - F_2) = 0$, which by Theorem 4.6 implies that $F_1 - F_2 = \nabla^S v$ for some v, whence $F_1 = F_2$, since div $\left(\nabla^S v\right) = 0$ in Ω with $(\nabla^S v)N = 0$ on $\partial\Omega$.

5. The intrinsic approach with a anholonomic curvilinear frame: statement of the main results

Definition 5.1 (Equivalence class)**.** *We write $u \doteq u_0$ on Γ_0 to mean that there exists a rigid displacement $r \in \mathcal{R}(\Gamma_0)$ in Γ_0, such that $u - r = u_0$.*

Lemma 5.2. *The following conditions are equivalent:*

(i) $e \in \mathscr{C}^1(\bar{\Omega}) \cap \mathcal{H}^2_{\Gamma_0;\mathrm{comp}}(\Omega)$,

(ii) *there exists a unique $u \in C^2(\bar{\Omega}) \cap H^3(\Omega, \mathbb{R}^3)$ such that $e = \nabla^S u$ and $u = 0$ on Γ_0. It also holds*

$$u \doteq 0 \text{ on } \Gamma_0 \iff e = 0 = \mathrm{Curl}^{\mathrm{t}}\, e \times N \text{ on } \Gamma_0.$$

Proof. If $e \in \mathscr{C}^1(\bar{\Omega}) \cap \mathcal{H}^2_{\Gamma_0;\mathrm{comp}}(\Omega)$, then inc $e = 0$ and by Theorems 4.6 and 4.7, there exists a unique $u \in C^2(\bar{\Omega}) \cap H^3(\Omega, \mathbb{R}^3)$ such that $e = \nabla^S u$ and $u = 0$ on Γ_0. Moreover, $e = 0 = \mathrm{Curl}^{\mathrm{t}}\, e \times N$ on Γ_0 and hence Theorem 4.3 entails that $\gamma^\sharp(e) = \rho^\sharp(e) = 0$. Hence Theorem 3.4 yields $u \doteq 0$ on Γ_0. Now, $u \in \mathcal{R}(\Gamma_0)$ implies by (3.40) and Theorem 4.3 that $e = 0 = \mathrm{Curl}^{\mathrm{t}}\, e \times N$ on Γ_0. □

Now, for $e \in \mathcal{H}_{\Gamma_0;\mathrm{comp}}(\Omega)$ Theorem 4.6 yields $e = \nabla^S u$, where uniqueness of u follows as soon as $\mathcal{H}^1(\Gamma_0) > 0$ [4]. Moreover, Corollary 4.5 provides an extension sense to the traces, since from (4.16) it follows that $e \in H^{-1/2}(\partial\Omega)$ and from (4.17) that $\mathrm{Curl}^{\mathrm{t}}\, e \in H^{-3/2}(\partial\Omega)$. In particular for any $e \in \mathcal{H}_{\Gamma_0;\mathrm{comp}}(\Omega)$ there exists a sequence $e_n \in \mathscr{C}^1(\bar{\Omega}) \cap \mathcal{H}^2_{\Gamma_0;\mathrm{comp}}(\Omega)$ such that $e_n \to e$ strongly in $L^2(\Omega, \mathbb{S}^3)$. Of course by Korn's inequality, it also holds $u_n \to u$ strongly in $H^1(\Omega, \mathbb{R}^3)$.

By the above considerations (in particular the weak traces of Section 4.2) and classical density arguments, we are now in position to state and prove a general form of Lemma 5.2. The following theorem is a restatement of Theorem 3.4 without appealing to the change of metric and curvature tensors as in the original version of [5], rather by letting e belong to a specific function space.

Theorem 5.3 (Intrinsic version of the homogeneous condition)**.** *The following conditions are equivalent:*

(i) $e \in \mathcal{H}_{\Gamma_0;\mathrm{comp}}(\Omega)$,

(ii) *there exists a unique $u \in H^1(\Omega)$ such that $e = \nabla^S u$ and $u = 0$ on Γ_0. It also holds*

$$u \doteq 0 \text{ on } \Gamma_0 \iff \bar{\mathscr{D}}_{1,0}(e) = \bar{\mathscr{D}}_{2,0}(e) = 0,$$

where $\mathscr{D}_i$ are the boundary operators given by (4.18) and (4.19) for $i = 1$ and $i = 2$, respectively.

Let us remark that the non-homogeneous problem could be considered in this setting, too. In contrast with Theorem 5.4 it is not clear how the nonhomogeneous boundary condition may be handled by means of the change of metric and curvature tensors as in the original version of [5]. Within our formalism, it is immediate, as stated in the following result.

Theorem 5.4 (Intrinsic version of the non-homogeneous condition)**.** *Let $u_0 \in H^{3/2}(\Gamma_0, \mathbb{R}^3)$. If $e \in \mathcal{H}_{\mathrm{comp}}(\Omega)$, there exists a unique $u \in H^1(\Omega)$ such that $e = \nabla^S u$ and $u = u_0$ on Γ_0. It also holds*

$$u \doteq u_0 \text{ on } \Gamma_0 \iff \bar{\mathscr{D}}_{1,0}(e - \nabla^S \hat{u}) = \bar{\mathscr{D}}_{2,0}(e - \nabla^S \hat{u}) = 0,$$

where $\hat{u} = \mathscr{L}(u_0)$ is a H^1-boundary lifting of u_0.

Proof. The first part of the statement follows from Saint-Venant compatibility condition and Beltrami decomposition in L^2 (Theorems 4.6 and 4.7) where the rigid displacement is fixed by the condition $u = u_0$ on Γ_0. The second part is an obvious consequence of Theorem 5.3. □

Definition 5.5. *We introduce that following quotient space with respect to the equivalence relation of Definition 5.1:*

$$\dot{H}^1_{\Gamma_0}(\Omega, \mathbb{R}^3) = H^1_{\Gamma_0}(\Omega, \mathbb{R}^3)/\mathcal{R}(\Gamma_0) = \{u \in H^1(\Omega, \mathbb{R}^3) : u \doteq 0 \text{ on } \Gamma_0\}.$$

Corollary 5.6. *The map*

$$\mathcal{F}^\star : \mathcal{H}_{\Gamma_0;\mathrm{comp}}(\Omega) \to \dot{H}^1_{\Gamma_0}(\Omega, \mathbb{R}^3) : e \longmapsto \mathcal{F}^\star(e) = u \text{ s.t. } e = \nabla^S u \tag{5.1}$$

is well defined, linear and continuous with respect to the L^2-norm.

Proof. Well-definedness and linearity follow from Theorem 5.3, and continuity from Korn inequality [14]. □

It is now obvious that the strong form of linearized elasticity may be rewritten as a variational problem in terms of the compatible strain $e = e(u) = \nabla^S u$.

Theorem 5.7 (Intrinsic version of linearized elasticity)**.** *The variational problem*

$$\inf_{e \in \mathcal{H}_{\Gamma_0;\mathrm{comp}}(\Omega)} \mathscr{E}(e) = \frac{1}{2}\int_\Omega \mathbb{C}^{-1} e \cdot e dx - \int_\Omega f \cdot F^\star(e) dx - \int_{\partial\Omega \setminus \Gamma_0} g \cdot F^\star(e) dS(x),$$

achieves its minimum $e^\star$, which satisfies the strong form

$$\begin{cases} -\operatorname{div}\left(\mathbb{C}^{-1} e^\star\right) = f & \text{in} & \Omega \\ e^\star = \operatorname{Curl}^{\mathrm{t}} e^\star \times N = 0 & \text{on} & \Gamma_0 \\ \left(\mathbb{C}^{-1} e^\star\right) N = g & \text{on} & \partial\Omega \setminus \Gamma_0 \end{cases},$$

where the traces are intended in a weak sense. Furthermore $u^\star := F^\star(e^\star)$ satisfies

$$\begin{cases} -\operatorname{div}\left(\mathbb{C}^{-1} \nabla^S u^\star\right) = f & \text{in} & \Omega \\ u^\star \doteq 0 & \text{on} & \Gamma_0 \\ \left(\mathbb{C}^{-1} \nabla^S u^\star\right) N = g & \text{on} & \partial\Omega \setminus \Gamma_0 \end{cases}.$$

Proof. By Korn inequality, it is easily computed that $0 > \mathscr{E}(e) > C\|e\|_{L^2} - \beta$ for some $C > 0$ and $\beta \geq 0$. Existence follows by the direct method, since $e \mapsto \mathscr{E}(e)$ is continuous by Corollary 5.6. □

Remark that the non-homogeneous case can also be written as a variational problem in e, by a simple boundary lifting, and change of variables.

Theorem 5.7 is the counterpart of Theorems 7.1 and 7.2. in [5]. However, instead of a condition on the variations of metric curvature, which we believe is unease to give a clear phiscal meaning, here we give a condition on the strain and on the Frank tensor on Γ_0, where both bear a precise physical meaning. Moreover, our formalism also allows one to write the problem in a variationla manner, since the boundary condition on the compatible strain is included in the function space $\mathcal{H}_{\Gamma_0;\mathrm{comp}}(\Omega)$.

6. Discussion and concluding remarks

6.1. **Application to multiscale analysis of dislocations.** A mesoscopic dislocation is a loop $\mathcal{L}$ in a crystal Ω which renders the strain field singular, because it implies a behaviour of the strain $\mathscr{O}(1/d(\cdot, \mathcal{L}))$, with $d(x, \mathcal{L})$ the distance from $x \in \Omega$ to $\mathcal{L}$. This is due to the constrain $\oint_{C_\mathcal{L}} \nabla u d\mathcal{H}^1 = B$, where B is the jump of the displacement vector, called the Burgers vector, constant on $\mathcal{L}$ and $C_\mathcal{L}$, a circuit around the line $\mathcal{L}$. Furthermore, ∇u is the absolutely continuous part of the distributional derivative $Du = \nabla u \mathscr{L}^3 + B \otimes N \mathcal{H}^2_{S_\mathcal{L}}$, with $S_\mathcal{L}$ a surface enclosed by $C_\mathcal{L}$ [16]. By Stokes theorem, one has $\operatorname{Curl} \nabla u = -\Lambda^T_\mathcal{L} = B \otimes \tau \mathcal{H}^1_\mathcal{L}$ which implies (as proved in [20]) that

$$-\operatorname{inc} e = \operatorname{Curl}\,(\Lambda_\mathcal{L} - \frac{\mathbb{I}_2}{2} \operatorname{tr} \Lambda_\mathcal{L}),$$

where $e := \nabla^S u \in L^1(\Omega)$ is the symmetric incompatible strain. As a consequence of the strain being $\mathscr{O}(1/d(\cdot, \mathcal{L}))$ the potential energy

$$\mathscr{E}(e) := \frac{1}{2}\int_\Omega \mathbb{C}^{-1} e \cdot e dx - \int_\Omega f \cdot F^\star(e) dx - \int_{\partial\Omega \setminus \Gamma_0} g \cdot F^\star(e) dS(x)$$

is unbounded, because so is the stored elastic energy

$$\mathscr{W}(e) := \frac{1}{2}\int_\Omega \mathbb{C}^{-1} e \cdot e dx.$$

Let $\mathcal{L}_\varepsilon$ be a finite family of dilute dislocations, whose number is bounded by N_ε. Let e_ε be a symmetric strain satisfying $\mathrm{inc}\, e_\varepsilon = \Lambda_{\mathcal{L}_\varepsilon}$ in Ω and $\hat{e}_\varepsilon$ their cut-off, i.e. $\hat{e}_\varepsilon = 0$ in a tubular neigbourhood of $\mathcal{L}_\varepsilon$ of a certain radius dependent on ε. Let us consider the rescaled stored elastic energy

$$\mathscr{W}_\varepsilon(\hat{e}_\varepsilon;\Lambda_{\mathcal{L}_\varepsilon}) = \frac{1}{N_\varepsilon|\log\varepsilon|}\int_\Omega \frac{1}{2}\mathbb{C}^{-1}\hat{e}_\varepsilon\cdot\hat{e}_\varepsilon dx,$$

We wonder whether the functional $\mathscr{W}_\varepsilon$ do converge in an appropriate sense, as for instance of Gamma-convergence [6]. A crucial step when dealing with Gamma-convergence, is to determinate the topology involved, and hence to know the functional spaces inherent to the problem. Here, $\Lambda_{\mathcal{L}_\varepsilon}$ is a bounded Radon measure, and the strains $\hat{e}_\varepsilon$ belong to $L^2(\Omega,\mathbb{S}^3)$. Thanks to our formalism we could consider the case in which the displacement is prescribed on Γ_0, as in Therorem 5.7, that is, we take $\hat{e}_\varepsilon \in \mathcal{H}_{\Gamma_0;\mathrm{comp}}(\Omega) \subset L^2(\Omega,\mathbb{S}^3)$. The problem is thus to compute

$$\Gamma - \lim_{\varepsilon\to 0}\mathscr{W}_\varepsilon(\hat{e}_\varepsilon;\Lambda_{\mathcal{L}_\varepsilon}).$$

There are good reasons to believe (but is hard to prove, and will be the aim of a future work) that this limit writes as

$$\mathscr{W}(\bar{e}) + \int_\Omega \varphi(\frac{d\mu}{d|\mu|})d|\mu|,$$

where $\frac{\hat{e}_\varepsilon}{\sqrt{N_\varepsilon|\log\varepsilon|}} \to \bar{e} \in \mathcal{H}_{\Gamma_0;\mathrm{comp}}(\Omega)$ weakly in L^2, and $\Lambda_{\mathcal{L}_\varepsilon} \to \mu$ in the Radon measure sense, and for some φ to be determined. Now, let $\Lambda_{\mathcal{L}}$ be fixed and set

$$\mathscr{E}_\varepsilon(\hat{e}_\varepsilon,\Lambda_{\mathcal{L}}) := \mathscr{W}_\varepsilon(\hat{e}_\varepsilon;\Lambda_{\mathcal{L}}) - \int_\Omega f\cdot F^\star(\hat{e}_\varepsilon)dx - \int_{\partial\Omega\setminus\Gamma_0} g\cdot F^\star(\hat{e}_\varepsilon)dS(x).$$

Then the above postulated Gamma-convergence result together with Corollary 5.6 and Theorem 5.7 yield

$$\Gamma - \lim_{\varepsilon\to 0}\left\{\inf_{\substack{\hat{e}_\varepsilon\in\mathcal{H}_{\Gamma_0;\mathrm{comp}}(\Omega)\\ \mathrm{inc}\ e=\Lambda_{\mathcal{L}}}}\mathscr{E}_\varepsilon(\hat{e}_\varepsilon;\Lambda_{\mathcal{L}})\right\} = \mathscr{E}(e^\star) + \int_{\mathcal{L}}\varphi(B\otimes\tau)d\mathcal{H}^1,$$

where φ is interpreted as a line-tension functional accounting for the core regularization of the dislocations, i.e., related to the self energy of the dislocation network.

Summarizing, our formalism of the intrinsic approach will allow us address the aforementioned homogenization problem, where one passes from a singular elasticity problem at the mesoscopic scale, to a regularized macroscopic elasto-plastic model, where the macroscopic plasticity is modelled by means of the limit measure μ. Note that it is important in dislocation modeling to be able to consider a complete boundary-value problem of mixed type, since typical crystals, in particular in industrial crystal growth processes, show force-free portion of their boundary together with interfaces subjected to an imposed displacement.

6.2. **General conclusion.** The motivation for this work was the study of dislocations where the displacement must be replaced by the strain as model variable. This work represents the first step towards a systematic use of the Frank tensor in various contexts, and in particular in the study of dislocations, where it most naturaly appears in the form of its curl as the incompatibility tensor, that is, by Kröner formula [12], as a measure of the dislocation density in the body:

$$\mathrm{inc}\,\epsilon = \mathrm{Curl}\,\kappa,$$

with κ the contortion tensor as related to the dislocation density Λ by $\kappa := \Lambda - \frac{\mathbb{I}_2}{2}\mathrm{tr}\,\Lambda$, and ϵ the elastic strain, related to the Cauchy stress σ by the constitutive law $\epsilon = \mathbb{C}\sigma$. For this reason, this work will permit various contributions in dislocation modelling, in particular for deaqling with homogenization, as briefly exposed in Section 6.1.

The original intrinsic formulation by Ph. Ciarlet and C. Mardare is also at the origin of the present work. The authors have in mind various applications in shell and plate theories, and the definitive impact of such a novel presentation will certainly appear clear in the near future. Therefore, also the role of the Frank tensor, related to the rotation gradient,

$$\mathrm{Curl}^{\mathrm{t}}\,\epsilon = \nabla w,$$

will presumably play a role in variational formulations of low-dimensional, membrane theories.

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