Logic and algebraic varieties in non-archimedean fields

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Encontros Abertos do CMAFcIO
10/09/2020

- Joint with: P. Cubides and Y. Ye;

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- Builds on:
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- Builds on:
  - Hrushovski and Loeser approach to Berkovich;
  - ideas from previews work on cohomology (E. with Peatfield, Jones and Prelli; Delfs)

Classical setting

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Work in

$$(\mathbb{C}, 0, 1, +, \cdot)$$

an algebraically closed field (ACF). Want to understand quasi-projective varieties, e.g. (locally of the form)

$$V = \{x \in \mathbb{C}^n : f_1(x) = \ldots = f_l(x) = 0\}$$

with $f_i \in \mathbb{C}[X_1, \ldots, X_n]$ polynomials.

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Moreover, $(\mathbb{C}, 0, 1, +, \cdot)$ is complete for the norm topology.

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Examples:

- If V is the projective space $\mathbb{P}^1$, then V^{an} is the Riemann sphere.
- If V is an elliptic curve (e.g. given by $y^2 = ax^3 + bx^2 + cx + d$), then V^{an} is a complex torus.

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How to distinguish these (or other such) objects in $\mathbb{C}^2$? Use cohomology

$$H^*_{\mathrm{sing}}(V^{\mathrm{an}}; \mathbb{C}) \simeq H^*(V^{\mathrm{an}}; \mathbb{C}) \simeq H^*_{dR}(V^{\mathrm{an}})$$

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- $H^p(V^{\mathrm{an}}; L)$ are finitely generated;
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Note: The same holds for cohomology with compact supports

$$H_c^*(V^{\mathrm{an}}; L)$$

Grothendieck setting

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What happens if we work in an arbitrary algebraically closed field (ACF)

$$(K, 0, 1, +, \cdot)$$

and consider quasi-projective varieties, e.g. (locally of the form)

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There is no norm! Alternatives:

- equip V with the Zariski topology … but we get very few open subsets…

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  - Replace V by the associated scheme, e.g. (locally of the form)

    $$\mathrm{Spec}(K[V]) = \{x : x \text{ is a prime ideal of } K[V]\}$$

    where $K[V] = K[X_1, \ldots, X_n]/I[V]$ and $I[V] = \langle f_1, \ldots, f_l \rangle$
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Note: $V \subseteq \mathrm{Spec}(K[V])$ identifying $(x_1, \ldots, x_n)$ with the maximal ideal $\langle X_1 - x_1, \ldots, X_n - x_n \rangle$.

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... and, if V is a scheme as above then, for étale cohomology we have:

(B)

- $H_c^p(V_{et}; L)$ are finite;
- $H^p(V_{et}; L) = 0$ for $p > 2\dim V$;
- If K' is an algebraically closed extension of K, then

$$H_c^*(V_{et}; L) \simeq H_c^*(V_{K',et}; L)$$

- (Artin) If $K = \mathbb{C}$, then

$$H_c^*(V_{et}; L) \simeq H_c^*(V^{\mathrm{an}}; L) \text{ and } H^*(V_{et}; L) \simeq H^*(V^{\mathrm{an}}; L)$$

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Note: finiteness and invariance holds without supports if $|L| \neq 0$ in K.

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Now work in an arbitrary ACF

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Assume also that K is a complete for topology of this norm.

Berkovich setting ...

Examples:

- Any ACF K with the trivial norm

$$|x| = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{otherwise} \end{cases}$$

- $\mathbb{C}_p$ the completion of the algebraic closure of $\mathbb{Q}_p$ - the completion of $\mathbb{Q}$ with the p-adic norm:

$$|x|_p = p^{-a} \qquad \text{if } x = \frac{p^a r}{s}$$

with p, r, s relatively prime.

...

- $\mathbb{C}\{\{t\}\}$ the field of Puiseaux series

$$f = \sum_{i=k}^{+\infty} a_i t^{\frac{i}{n}} \quad \text{some } n \neq 0 \text{ and } k \in \mathbb{Z}$$

i.e., the completion of the field of Laurent series $\mathbb{C}((t))$,

$$f = \sum_{i=k}^{+\infty} a_i t^i \quad \text{some } k \in \mathbb{Z}$$

(the fraction field of the ring of formal power series $\mathbb{C}[[t]]$) with the t-adic norm:

$$|f|_t = e^{-m} \qquad \text{if } f = t^m \left(\sum_{i=0}^{+\infty} a_i t^i\right)$$

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  - Replace V by

$$V^{\mathrm{an}} = \{(x, |\ |_x) : x \in \mathrm{Spec}(K[V]) \text{ and } |\ |_x : K(x) \to \mathbb{R}_{\geqslant 0}\}$$

    where $|\ |_x$ extends $|\ | : K \to \mathbb{R}_{\geqslant 0}$.
  - Put on V^{an} the coarsest topology making all $x \mapsto |f|_x$ continuous for all $f \in \mathcal{O}_V(U)$ and all $U \subseteq V$ Zariski open.

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Note: This new V^{an} is a nice locally compact topological space.

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(A) (Under extra assumptions) V^{an} admits a deformation retraction to a finite simplicial complex.

(B)

- $H^p_c(V^{\mathrm{an}}; L)$ are finitely generated;
- $H^p(V^{\mathrm{an}}; L) = 0$ for $p > \dim V$;
- There is a finite Galois extension $K \leqslant K'$ such that for any non-archimeadean field extension $K' \leqslant K''$ we have

$$H^*_c(V^{\mathrm{an}} \widehat{\otimes} K'; L) \simeq H^*_c(V^{\mathrm{an}} \widehat{\otimes} K''; L)$$

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- $\mathrm{val}(f \cdot g) = \mathrm{val}(f) + \mathrm{val}(g)$;
- $\mathrm{val}(f + g) \geqslant \min\{\mathrm{val}(f), \mathrm{val}(g)\}$

where $\Gamma = (\Gamma, 0, +, <)$ is an ordered abelian (divisible) group and ∞ is such that $\gamma < \infty$ for all $\gamma \in \Gamma$.

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Note: If $\Gamma \subseteq \mathbb{R}$ i.e. valuations of rank one, then

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- Replace V by

$$\widehat{V} = \{x : x \text{ is a } K\text{-definable type on } V \text{ orthogonal to } \Gamma_\infty\}$$

- Put on $\widehat{V}$ the topology with pre-basis

$$\{x \in \widehat{U} : (\mathrm{val} \circ f)_*(x) \in I\}$$

with $U \subseteq V$ Zariski open, $f \in \mathcal{O}_V(U)$ and $I \subseteq \Gamma_\infty$ an open interval.

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(A') If K is a complete rank one valued field and $K^{\max}$ is a canonical maximally complete extension of K, then:

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(A') If K is a complete rank one valued field and $K^{\max}$ is a canonical maximally complete extension of K, then:

- $V^{\mathrm{an}} \widehat{\otimes} K^{\mathrm{an}}$ is canonically homeomorphic to $\widehat{V}(K^{\max})$;
- The pro-definable deformation retraction restricts to a deformation retraction of V^{an} to a definable subset of $\mathbb{R}_\infty$.

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- $H^p_c(\widehat{V}; L) = 0$ for $p > \dim V$;
- If K' is an algebraically closed valued field extension of K, then

$$H^*_c(\widehat{V}; L) \simeq H^*_c(\widehat{V}(K'); L)$$

- If K is a complete rank one valued field, then there is a finite Galois extension $K \leqslant K'$ such that for any $K' \leqslant K''$ we have

$$H^*_c(\widehat{V}(K^{\max}); L) \simeq H^*_c(\widehat{V}(K^{\max})_{\mathrm{top}}; L) \simeq H^*_c(V^{\mathrm{an}} \widehat{\otimes} K''; L)$$

(here L is a finitely generated module over a noetherian ring)

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- Extra work is needed to get finiteness and invariance without supports.
- ... in this new world a lot of other stuff is still to be done.

Thank you!