Some Moduli Spaces and their Invariants

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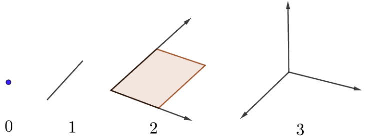
Encontros Abertos CMAFcIO, Sep. 2020

Outline

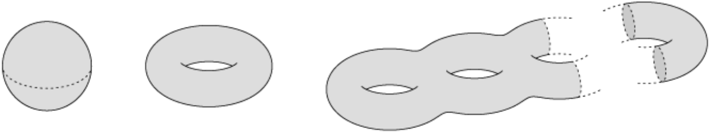
Famous Classification Problems (discrete invariants)

- Finite dimensional $\mathbb{R}$ or $\mathbb{C}$ vector spaces:



(isomorphism classes)

- Closed, orientable surfaces:



(homeomorphism classes)

Theorem (Classification of topological surfaces)

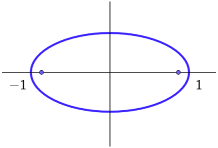
Every closed, orientable, topological surface is homeomorphic to a connected sum of $g \in \mathbb{N}$ tori.

Famous Classification Problems (continuous invariants)

- Conics in the plane:

$$ax^2 + 2bxy + cy^2 = 1$$

is an ellipse / hyperbola if $ac - b^2 > / ac - b^2 < 0$.



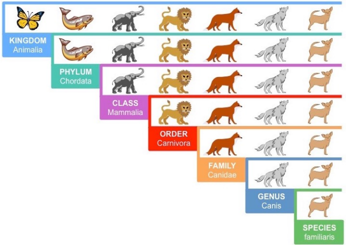
- Square matrices, up to conjugation:

$$M \simeq M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_k},$$

for a sequence $\lambda_1, \cdots, \lambda_k \in \mathbb{C}$, and each M_λ a direct sum of Jordan blocks of eigenvalue λ.

Steps for a Classification Problem

1. Define our *Universe* - its *elements*
2. Observe common properties; *Distinguish/Relate elements*
3. *List / Parametrize* all elements (up to equivalence)
   Discrete / Continuous

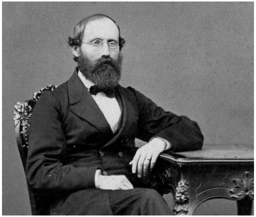


hydrogen H 1							helium He 2
lithium Li 3	beryllium Be 4	boron B 5	carbon C 6	nitrogen N 7	oxygen O 8	fluorine F 9	neon Ne 10
sodium Na 11	magnesium Mg 12	aluminium Al 13	silicium Si 14	phosporus P 15	sulfur S 16	chlorine Cl 17	argon Ar 18
potassium K 19	calcium Ca 20					bromine Br 35	krypton Kr 36
rubidium Rb 37	strontium Sr 38					iodine I 53	xenon Xe 54
cesium Cs 55	barium Ba 56						radon Rn 86
francium Fr 87	radium Ra 88						

Classification Problems in **Mathematics**:

1. Universe: vector spaces; groups; topological spaces, manifolds, functions ...
2. Equivalences: Morphisms between objects; isomorphisms (homeo/diffeomorphism, ...)
3. List / Parametrize / "Geometrize": Invariants / Moduli spaces

What is a Moduli space?

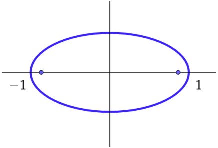


Bernhard Riemann, 1857: "*und es haengt von $3g - 3$ stetig veraenderlichen Groessen ab, welche die Moduln dieser Klasse genannt werden sollen*".

(and it depends on $3g - 3$ varying quantities, which are called **the moduli** of this class)

A *moduli space* is a space ***parametrizing*** a given class of geometric objects (up to equivalences), all belonging to the same "family".

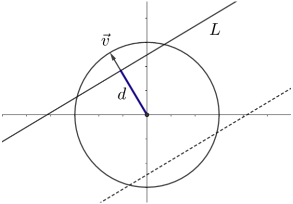
Toy Example: Ellipses in the plane:
$ax^2 + 2bxy + cy^2 = 1$, with $ac - b^2 > 0$.
The *invariants* $\Delta := ac - b^2$ and $T := a + c$ completely classify ellipses, up to *euclidean motions*.
The **moduli** are (T, Δ) subject to $0 < \Delta < \frac{T^2}{4}$.



Toy Examples "from ancient Greece"

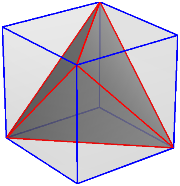
Problem 1: Lines in $\mathbb{R}^n$ through 0:
The *projective space* $\mathbb{P}(\mathbb{R}^n) \equiv \mathbb{RP}^{n-1} := (\mathbb{R}^n \setminus \{0\})/\mathbb{R}^*$
Variation: Lines in $\mathbb{C}^n$ through 0 are parametrized by $\mathbb{CP}^{n-1} := (\mathbb{C}^n \setminus \{0\})/\mathbb{C}^*$. Action: $(z_1, \cdots .z_n) \sim (\lambda z_1, \cdots, \lambda z_n)$

Problem 2: Classify lines in the plane



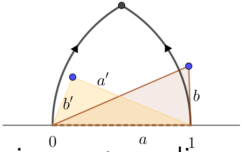
Problem 3: Triangles in $\mathbb{R}^2$:

$$\begin{array}{ccccc} 0 & < & a & < & b+c \\ 0 & < & b & < & a+c \\ 0 & < & c & < & a+b \end{array}$$

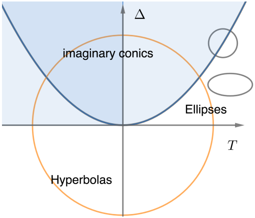


Toy Examples II

Problem 4: Triangles in $\mathbb{R}^2$, up to scaling:



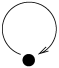
Problem 5: Invariants of conics, up to scaling:



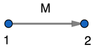
$$(T,\Delta)\sim(kT,k^2\Delta),\quad k\in\mathbb{C}^*$$
$$T=a+c,\quad \Delta=ac-b^2,$$

moduli space is $\mathbb{CP}^1_{1,2}$

Problem 6: Invariants of Matrices (quiver representations!):



$\mathrm{Sym}^n(\mathbb{C})=\mathbb{C}^n/S_n$



$\{0,\cdots,n\}$

The "original" moduli space

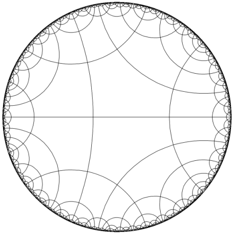
Closed, orientable surfaces admit complex structures: **Riemann surfaces**. And non-equivalent complex structures can be parametrized via a moduli space.

Theorem (Poincaré, Koebe, 1904 - compact Riemann surfaces)

Every closed, orientable surface X admits a constant curvature metric. Moreover, if $g > 1$ every such X is of the form:

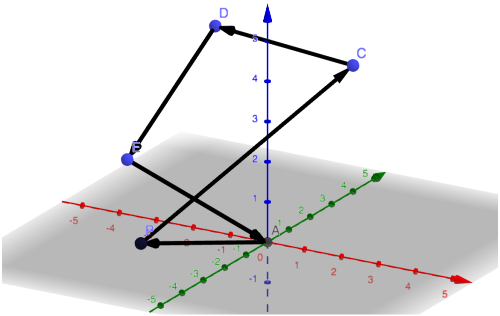
$$X = \mathbb{H}/\Gamma,$$

with $\Gamma \cong \pi_1 X \subset PSL_2\mathbb{R}$ discrete, acting by Möbius transformations



$$\mathcal{M} \simeq \hom^*(\pi_1 X, PSL_2\mathbb{R})/PSL_2\mathbb{R}$$
$$\dim \mathcal{M} = 3g - 3$$

The moduli space of spatial polygons



Fix lengths $\alpha_1, \cdots, \alpha_m \in \mathbb{R}_{>0}$ and let

$$Y^m_\alpha := \{(v_1, \cdots, v_r) \in S^2(\alpha_1)\times\cdots\times S^2(\alpha_m) \mid v_1+v_2+\cdots+v_m = 0\}$$

Moduli space of polygons with length vector α

$$Pol_m(\alpha) = Y^m_\alpha / SO(3)$$

This moduli space is compact, but generally singular.

Topological invariants - a brief tour

Let X be a compact finite dimensional manifold.

Facts: (1) The **cohomology groups** $H^k(X, \mathbb{C})$ of X can be computed from "triagulation data".
(2) $H^k(X) \equiv H^k(X, \mathbb{C})$ do not depend on triangulation, only on global topology!
(3) Each kth cohomology can be regarded as a functor: $Man \to Vect$.

We obtain a sequence of ***Betti numbers:***
$b_0 = \dim H^0(X), \quad b_1 = \dim H^1(X), \quad \cdots, b_n = \dim H^n(X).$
For compact connected, orientable manifolds of (real) dimension n:

$$b_0 = b_n = 1$$

Examples: The spheres $b_k(S^n) = 0$ for $k \neq 0, n$.
The tori $b_k(T^n) = \binom{n}{k}$.

Poincaré polynomial

Definition

With $b_k(X) = \dim_{\mathbb{C}} H^k(X, \mathbb{C})$, the Poincaré polynomial of X is:

$$P_t(X) = \sum_{k \geq 0} b_k(X)\ t^k,$$

Integer / Polynomial invariants associated to geometric objects:

Object M	Euler char. $\chi(M)$	Poincaré polynomial $P_t(M)$
$\mathbb{R}^n$	1	t^n
Σ_g	$2 - 2g$	$1 + 2gt + t^2$
S^n	$1 + (-1)^n$	$1 + t^n$
$\mathbb{CP}^n$	$n + 1$	$1 + t^2 + \cdots + t^{2n}$
$Rep(Q)$	?	?
$Pol_m(\alpha)$	?	?

Examples: invariants P / μ for some moduli spaces

Theorem (Gothen, '94)

Let $\mathcal{M}$ be the moduli space of rank 3 Higgs bundles over a Riemann surface of genus 2 and fixed degree 1 determinant. Then: $P_t(\mathcal{M}) = 1 + 3t^2 + 20t^3 + 54t^4 + 416t^5 + 572t^6 + 376t^7 + 117t^8 + 32t^9 + 47t^{10} + 56t^{11} + 42t^{12} + 28t^{13} + 16t^{14} + 8t^{15} + 3t^{16}$.

The cohomology of a quasi-projective algebraic variety X (of dimension d) decomposes into "Hodge pieces" of dimensions $h^{k,p,q}(X)$, $k, p, q \in \{0, \cdots, 2d\}$. Mixed Hodge polynomial:

$$\mu(X; t, u, v) := \sum_{k,p,q} h^{k,p,q}(X) t^k u^p v^q,$$

Note: $P_t(X) = \mu(X, t, 1, 1)$, when X smooth and projective.

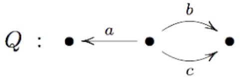
Theorem (F-Silva, '18)

Let $\mathcal{M}$ be the moduli space of representations of $\mathbb{Z}^r$ into G. Then:

$$\mu(X; t, u, v) = \frac{1}{|W|} \sum_{\sigma \in W} \det(I + tuvM_\sigma)^r.$$

Star shaped quivers

Consider a quiver (= a bunch of arrows):

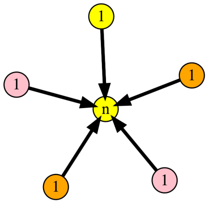


$$Q = (V, A), \qquad V = \{0, 1, \cdots, m\}, \qquad A \subset V \times V.$$

We have the generalized Endomorphism space:

$$End(Q) := \oplus_{k=0}^{m} W_k, \qquad \dim W_k = d_k$$

Automorphism group: $G(Q) := \times_{k=1}^{m} GL(W_k)$.
Representation space: $Rep(Q)_\alpha = End(Q) /\!\!/_\alpha G(Q)$



$$Rep_{m,n}(Q)_\alpha := (\mathbb{C}^n)^m /\!\!/_\alpha [GL(n, \mathbb{C}) \times (\mathbb{C}^*)^m]$$
$$\dim_{\mathbb{C}} Rep_{m,n}(Q)_\alpha = (n-1)(m-n-1)$$
$$d = (n, 1, \cdots, 1)$$

Closely related with **parabolic bundles** over $\mathbb{CP}^1$,
Other quivers (with relations) - **Calabi-Yau**'s in *string theory*!

Equivalence: Algebraic $\longleftrightarrow$ Symplectic quotients

Let $n = 2$, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m) \in \mathbb{R}^m_{>0}$. Then
$End(Q) = \{m \text{ vectors } w_i \in \mathbb{C}^2\}$.
Define $End_\alpha(Q) \subset End(Q)$ as those with $|w_i|^2 = \alpha_i$.

Moment map:

$\mu_\alpha : End_\alpha(Q) \to \mathfrak{su}(2)^* = \mathfrak{so}(3)^*, \qquad \mu_\alpha(A) := \sum_{i=1}^m (w_i w_i^*)_0$

Theorem (A special case of Kempf-Ness, Kirwan's Theorem)

There is a diffeomorphism:

$$Rep_{m,2}(Q)_\alpha = (\mathbb{CP}^1)^m /\!\!/_\alpha SL_2\mathbb{C} = \mu_\alpha^{-1}(0)/SU(2) = Y^m_\alpha / SO(3) = Pol_m(\alpha)$$

Example: Let $\alpha = (1,1,1,1,1)$ ("equal sided" pentagons in $\mathbb{R}^3$)

$$Pol_5(\alpha) = \mathbb{CP}^2 \# 4\overline{(\mathbb{CP}^2)},$$

(4 blow-ups of $\mathbb{CP}^2$) is a del Pezzo surface of degree 5 $\Rightarrow$
$P_t(Pol_5(\alpha)) = 1 + 5t^2 + t^4$.

The "moves" $Pol_m(\alpha) \leftrightarrow Pol_m(\alpha')$ can be obtained by *wall-crossing*.

Some references (please ask!)

Thank you!
Obrigado pela atenção!

- S. Mukai, *An Introduction to Invariants and Moduli*, Cambridge U. P, 2003
- F. Kirwan, *Cohomology of Quotients in Symplectic and Algebraic Geometry*, Princeton U. P, 1985
- A. Klyachko, *Spatial polygons and stable configurations of points in the projective line*, Alg. Geom. Appl, Aspects of Mathematics **25** 1994.
- H. Derksen, J. Weyman, *An introduction to quiver representations*, Amer. Math. Soc. 2017
- S. Y. Oudot, *Persistence Theory: From Quiver Representations to Data Analysis*, Amer. Math. Soc. 2017