# (Un)conventional approaches to the Riemann Hypothesis

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# The Riemann $\zeta$ function

## Definition

$\zeta : \mathbb{C} \to \mathbb{C}$ is the holomorphic function defined for $\Re(s) > 1$ by the (locally uniformly convergent) Dirichlet series

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## Theorem (Euler product)

*For* $\Re(s) > 1$,

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

# The functional equation (Riemann)

**Theorem (Functional equation)**

*For all $s \in \mathbb{C} \setminus \{1\}$*

$$\zeta(s) = 2^s \pi^{s-1} \sin(\pi s/2) \Gamma(1-s) \zeta(1-s),$$

*where $\Gamma$ is the (well-known) Gamma function.*

This is an identity between meromorphic functions which, carefully interpreted, allows for the meromorphic extension of $\zeta$ to the complex plane.

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Note: more general $\zeta$ functions may be constructed from Dirichlet series of multiplicative functions (e.g. characters), as $L-$functions. They satisfy an Euler product and functional equation.

# The zeros of the $\zeta$ function

The $\zeta$ function os zero-free on the half-plane $\Re(s) > 1$ and, from the functional equations, has simple zeros at $z = -2n$, $n \in \mathbb{N}$.
These are the *trivial* zeros.
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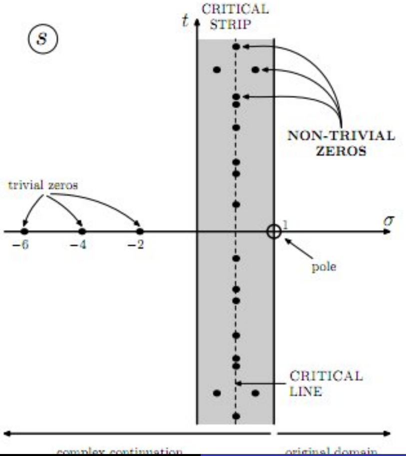
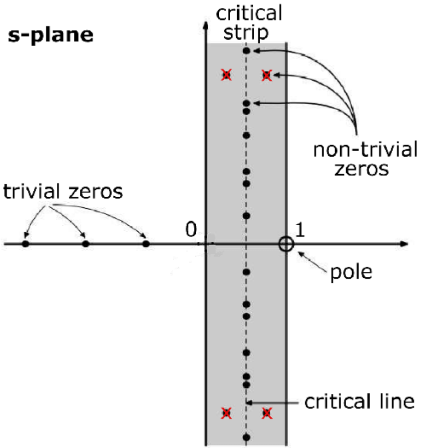


# The Riemann Hypothesis (RH)

**Conjecture (Riemann Hypothesis)**

All the non-trivial zeros of $\zeta$ lie on the critical line $\Re(z) = 1/2$.



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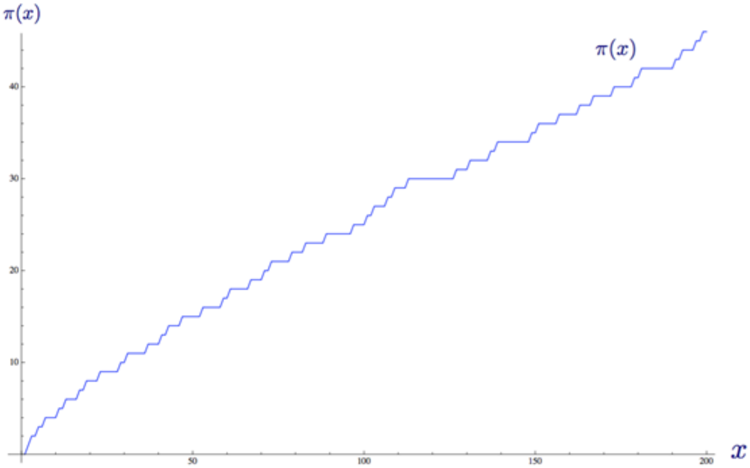
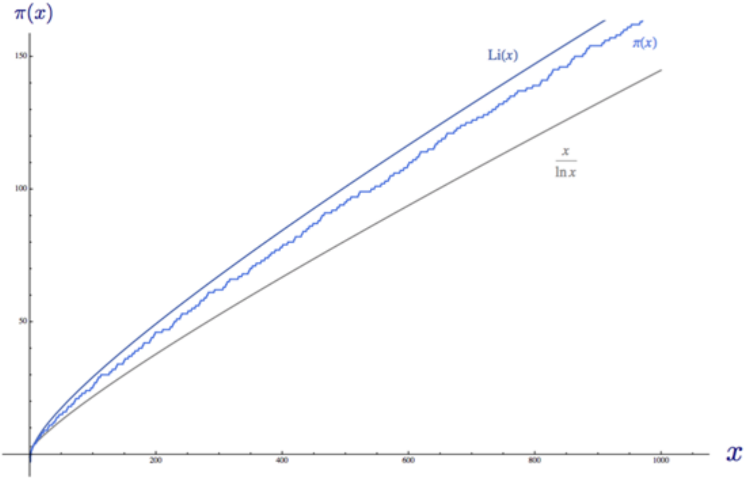
- Many clumps of billions of zeros up to $10^{24}$ lie on the critical strip.
- More than 5/12 of the zeros on the strip are on the critical line (2019).

# Where does RH come from?




The staircase of primes ($\pi(x)$ = number of primes $\leq x$).

# The Prime Number Theorem




The prime number theorem $(Li(x) = \int_2^x 1/\log t \, dt.)$

# Riemann's explicit formula

In his 1859 paper Riemann starts by defining an alternate prime counting function

$$\Pi(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \cdots + \frac{1}{n}\pi(x^{1/n}) + \cdots$$

from which $\pi(x)$ may be recovered by Möbius inversion

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**Theorem (Riemann's explicit formula)**

$$\Pi(x) = Li(x) - \sum_{\rho} Li(x^{\rho}) - \log 2 + \int_x^{+\infty} \frac{dt}{t(t^2-1)\log t},$$

*where the sum is over the zeros $\rho$ of the $\zeta$ function.*

# Riemann zeros and oscillatory behaviour

In Riemann's explicit formula

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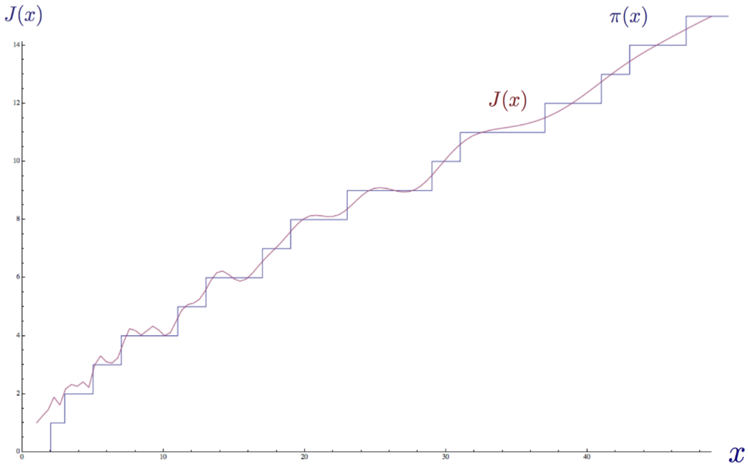
- the principal term $Li(x)$ is the smooth, monotonic asymptotic term from the PNT;
- in the second term, each term in the summation has an oscillatory nature since $\rho$ are strictly complex;
- the third term is constant and the fourth term vanishes as $x \to \infty$.

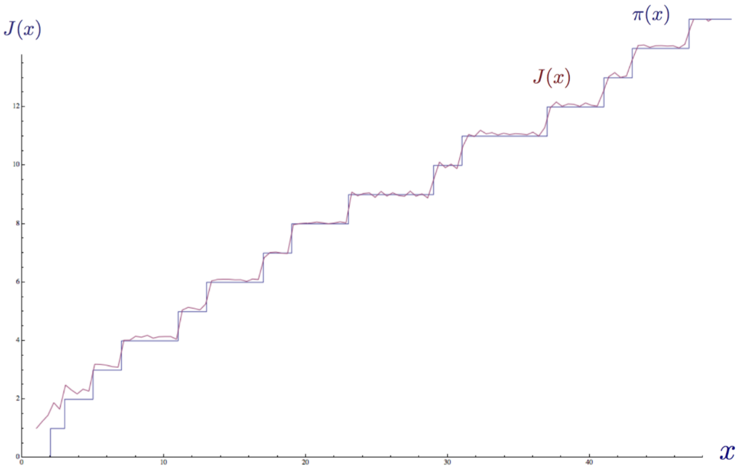
Thus the zeta zeros contribute with a "Fourier-like" series of oscillatory correction terms yielding an exact formula for the primes.

# Summing a series in zeta zeros, 1




The Riemann formula with 35 zeta zeros.

# Summing a series in zeta zeros, 2




The Riemann formula with 100 zeta zeros.

Convergence to Riemann's $\Pi(x)$ (animation).

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- Generalized RH (for $L$-functions) have an enormous wealth of consequences for other fields of Mathematics as well
- The RH appears in unexpected, apparenty unrelated contexts.

# Spectral interpretation of RH 1

By Fourier transforming the distributional density of (an appropriate version of) Riemann's counting function we obtain

$$F(t) = -\sum_{p^n} \frac{\log p}{p^{n/2}} \cos(t(\log(p^n))).$$

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This has its spectrum precisely at the zeros of the $\zeta$ function.

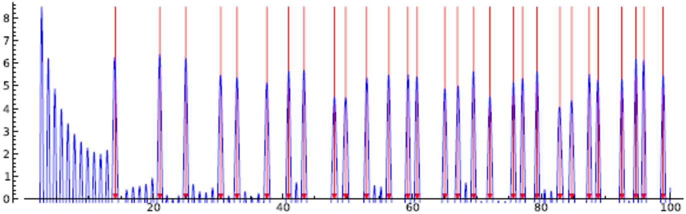


Figure 30.5: Plot of $-\sum_{p^n \leq 500} \frac{\log(p)}{p^{n/2}} \cos(t\log(p^n))$ with arrows pointing to the spectrum of the primes

(from Mazur and Stein)

# Spectral interpretation of RH 2

Conversely, subject to RH the Fourier-like series

$$G(s) = -\sum_i \cos((\log(s)\rho_i)),$$

where $\rho_i$ is the $i-$th zero of $\zeta$, converges to the corresponding distribution concentrated at the primes and prime powers.

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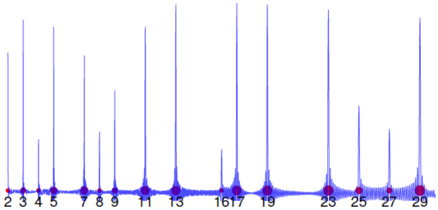


Figure 35.1: Illustration of $-\sum_{i=1}^{1000} \cos(\log(s)\theta_i)$, where $\theta_1 \sim 14.13, \ldots$ are the first 1000 contributions to the Riemann spectrum. The red dots are at the prime powers $p^n$, whose size is proportional to $\log(p)$.

(from Mazur and Stein)

This duality breaks down if RH is false.

# Random matrix theory

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H. Montgomery (1972): the pair correlation function between zeros of the zeta function behaves asymptotically as $\left(1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2\right)$.

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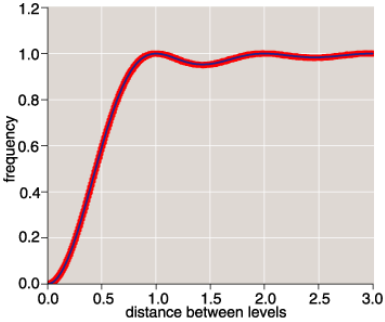


F. Dyson pointed out that this correlation is that of the eigenvalues of random Hermitian matrix theory...

...and of spectra of atomic nuclei, where the repulsion is of fermionic origin.


1-D distributions with different statistics, from Hayes.

In the 1990s, A. Odlyzko showed numerically that the zeta zero distribution follows to an incredible extent that of the Gaussian Unitary Ensemble (GUE).

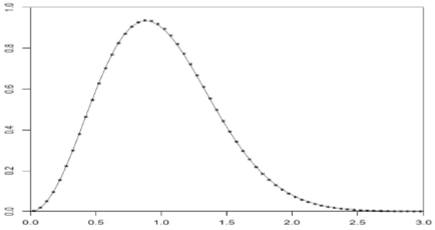


FIG. 3 – The distribution function of asymptotic gaps between eigenvalues ($\partial_s \det(\mathrm{Id} - K_{(0,s)})$) compared with the histogram of gaps between normalized $\zeta$ zeros, based on a billion zeros near $\#1.3 \cdot 10^{16}$ (by Odlyzko).

# Holomorphic PDFs, strip of holomorphy

$f$ is said to be a positive-definite function (PDF) if:

$$\sum_{j,k=1}^{n} f(z_j - \overline{z_k})\, \xi_j \overline{\xi_k} \geq 0$$

Example: extensions of characteristic functions from probability.
A real-analytic PDF $f : \mathbb{R} \to \mathbb{C}$ extends holomorphically to a maximal horizontal strip of the complex plane

$$\mathcal{S}_{\alpha,\beta} = \{z \in \mathbb{C} : -\alpha < \Im(z) < \beta\}$$

via the extension of the Bochner representation

$$f(z) = \int_{-\infty}^{+\infty} e^{itz}\, d\mu(t).$$

This strip is bounded by poles on the imaginary axis.

Analogously, we say $g$ is a co-PD function if

$$\sum_{j,k=1}^{n} f(z_j + \overline{z_k})\xi_j\overline{\xi_k} \geq 0.$$

**Theorem (BPS 2015)**

*A function $g$ defined on a vertical strip $T_{a,b}$) is a holomorphic co-PDF if and only if $g = \int_{-\infty}^{+\infty} e^{-zt} d\mu(t)$, where $\mu$ is an exponentially finite* **positive** *measure with respect to the interval $I = ]a, b[$. The measure $\mu$ is unique.*

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# co-PDF approaches to RH

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However, each of the functions

- $Z(s) = \zeta(s)/s$
- $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ (Riemann $\xi$ function)
- $\theta(s)$, Jacobi $\theta$ function

is holomorphic co-PDF on the critical strip, and their zeros coincide with those of the $\zeta$ function on the strip.

$\xi$ is a "symmetrized" version of zeta, introduced by Riemann himself, and satisfying the functional equation $\xi(s) = \xi(1-s)$.

# A co-PDF sufficient condition for RH

On the critical strip, for fixed $\sigma \in (0,1)$, $f(\sigma + it)$ is a PDF for any of the above functions. None of them can be an infinitely divisible (ID) PDF for any $\sigma$ (this would imply RH, but it is simply false). However:

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**Theorem (B-Paixão 2020)**

*If for $f$ equal to any of the functions $Z, \xi$ or $\theta$ it is true that $f(\sigma + it)$ is a quasi-ID PDF for every $\sigma \in (1/2, 1)$, then the Riemann Hypothesis is true.*

# And finally...

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THANK YOU
FOR YOUR ATTENTION!