The eigenvalues of the Robin Laplacian: a survey

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The Laplacian

Some prototypical partial differential equations (PDEs):

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad \text{(heat equation)}, \ k > 0$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \qquad \text{(wave equation)}, \ c > 0$$

(plus boundary and initial conditions) in an interval $I \times [0, \infty)$ where

$$\Delta u = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}$$

is the Laplacian (trace of the Hessian) in the $x = (x_1, \dots, x_d)$ (space) variables. Typical *Ansatz*: separation of variables / abstract Fourier series: solve the elliptic PDE

$$-\Delta u = \lambda u \qquad \text{in } \Omega$$

plus boundary conditions, then superpositions give the general solutions of the heat and wave equations.

The Laplacian

Solve/understand the *Helmholtz equation*

$$-\Delta u = \lambda u \qquad \text{in } \Omega \subset \mathbb{R}^d$$

subject to boundary conditions

- $u = 0$ on $\partial\Omega$: Dirichlet (first kind), fixed membrane/temperature
- $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$: Neumann (second kind), free membrane/perfect insulation (no flux)
- $\theta\frac{\partial u}{\partial \nu} + (1-\theta)u = 0$ on $\partial\Omega$: Robin (third kind), elastically supported membrane/imperfect insulation ($\theta \in (0,1)$)

Rewrite (and generalise) the Robin boundary condition:

$$\frac{\partial u}{\partial \nu} + \alpha u = 0 \qquad \text{on } \partial\Omega,$$

where $\alpha \in \mathbb{R}$, or $\mathbb{C}$, or $\alpha : \partial\Omega \to \mathbb{R}$ or $\mathbb{C}$ is a function.

The Laplacian with Robin boundary conditions

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega \subset \mathbb{R}^d \\ \frac{\partial u}{\partial \nu} + \alpha u &= 0 && \text{on } \partial\Omega \end{aligned}$$

- $\alpha = 0$: Neumann; "$\alpha = +\infty$" (and "$\alpha = -\infty$"?): Dirichlet
- Problem can't generally be solved explicitly (even Dirichlet and Neumann only for a few special domains)
- Today: keep things simple(r), $\alpha \in \mathbb{R}$ (mostly)

Goals

Understand how the solutions (eigenvalues λ and eigenfunctions u) depend on:

- The parameter α;
- The domain Ω and its "geometry".

Special case: 1D, dependence on $\alpha \in \mathbb{R}$

$$\begin{aligned} -u'' &= \lambda u \qquad \text{in } (0,1) \\ -u'(0) + \alpha u(0) &= 0 \\ u'(1) + \alpha u(1) &= 0 \end{aligned}$$

Eigenfunctions are linear combinations of $\sin(\sqrt{\lambda} x)$ and $\cos(\sqrt{\lambda} x)$, use boundary conditions to obtain:

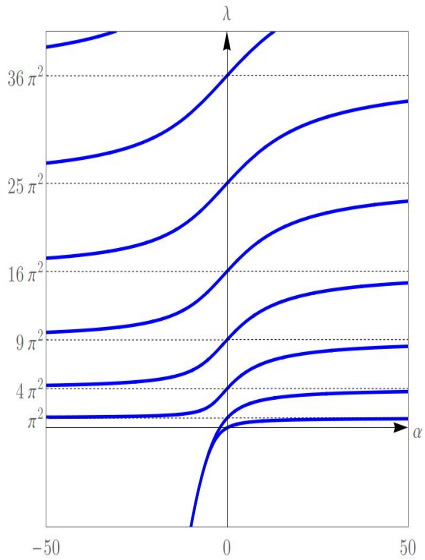
$$\lambda \text{ eigenvalue} \iff \alpha^2 + \alpha\sqrt{\lambda}\cot(\sqrt{\lambda}) - \lambda = 0.$$

Note: for each problem, i.e. each fixed α, there will be a sequence of eigenvalues

$$\lambda_1 \le \lambda_2 \le \lambda_3 \le \ldots \to +\infty$$

(use the spectral theorem).

Plot $\alpha^2 + \alpha\sqrt{\lambda}\cot(\sqrt{\lambda}) - \lambda = 0$



Observations:

- λ varies smoothly with α
- The eigenvalues are monotonically increasing in α
- $\alpha = 0$: Neumann, $\alpha \to +\infty$: convergence to Dirichlet from below
- $\alpha \to -\infty$: convergence to Dirichlet from above BUT $\exists$ divergent eigenvalues

The case of general $\Omega \subset \mathbb{R}^d$

$$
\begin{aligned}
-\Delta u &= \lambda u && \text{in } \Omega \subset \mathbb{R}^d\\
\frac{\partial u}{\partial \nu} + \alpha u &= 0 && \text{on } \partial\Omega,
\end{aligned}
$$

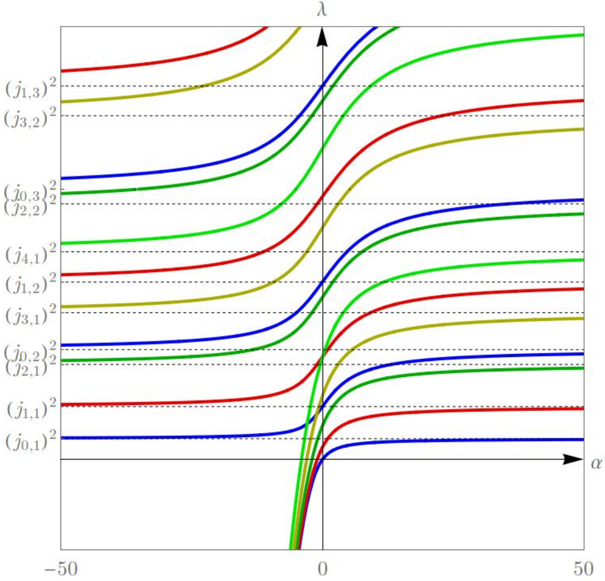
$\alpha \in \mathbb{R}$. Self-adjoint operator on $L^2(\Omega)$ with compact resolvent, eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \to +\infty$.

Theorem (folklore)

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with sufficiently smooth boundary and $\alpha \in \mathbb{R}$. Then

- Each eigenvalue λ_n is a piecewise analytic function of α (there may be crossings of curves);
- Each eigenvalue λ_n is a monot. increasing function of α;
- As $\alpha \to +\infty$, λ_n converges to the n-th eigenvalue of the Dirichlet Laplacian from below;
- As $\alpha \to -\infty$ $\exists$ *sequence* of eigenvalues diverging to $-\infty$.

Some of the eigencurves for the unit disk



The case of general $\Omega \subset \mathbb{R}^d$

Ingredients of the proof

- Smoothness: analytic perturbation theory of Kato.
- The weak form of the eigenvalue equation: (λ, u) eigenpair iff

$$\int_\Omega \nabla u \cdot \nabla v + \alpha \int_{\partial\Omega} uv = \lambda \int_\Omega uv$$

  for all test functions $v \in H^1(\Omega)$.
- Min-max characterisation of the eigenvalues, e.g.

$$\lambda_1 = \inf_{0 \neq u \in H^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 + \alpha \int_{\partial\Omega} u^2}{\int_\Omega u^2} = \inf_{0 \neq u \in H^1(\Omega)} \frac{\langle u, -\Delta u \rangle_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}}.$$

The asymptotic behaviour as $\alpha \to -\infty$

$$\lambda_1 = \inf_{0 \neq u \in H^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 + \alpha \int_{\partial\Omega} u^2}{\int_\Omega u^2}$$

Necessarily $\lambda_1 < 0$ if $\alpha < 0$, and in 1D we expect exponentials in place of trig functions ($-u'' = \lambda u$, $\lambda < 0$).

General principle

The divergent eigenvalues behave like $\lambda \sim -\alpha^2$ as $\alpha \to -\infty$. Intuitively: $u(x) = e^{\alpha x}$ eigenfunction of

$$-u''(x) = -\alpha^2 u(x) \qquad \text{in } (0, \infty)$$
$$-u'(0) + \alpha u(0) = 0.$$

On general domains there exists a sequence of eigenfunctions concentrating exponentially (like $e^{\alpha x}$) near the boundary.

The asymptotic behaviour as $\alpha \to -\infty$

Theorem (Test function argument, Giorgi–Smits 2007)

If $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain, then $\lambda_1 \leq -\alpha^2$ for all $\alpha < 0$.

Theorem (Lacey–Ockendon–Sabina 1998, Lou–Zhu 2004, Levitin–Parnovski 2008)

- If $\Omega \subset \mathbb{R}^d$ is C^1, then $\lambda_1 = -\alpha^2 + o(\alpha^2)$ as $\alpha \to -\infty$.
- If $\Omega \subset \mathbb{R}^2$ is piecewise smooth with "model corners", then $\lambda_1 = -C\alpha^2 + o(\alpha^2)$ for some $C \geq 1$ which is larger for "pointier" corners.

Theorem (Daners–K. 2010)

If $\Omega \subset \mathbb{R}^d$ is C^1, then for each $n \in \mathbb{N}$,

$$\lambda_n = -\alpha^2 + o(\alpha^2) \qquad \text{as } \alpha \to -\infty.$$

Since ca. 2013: More terms in the asymptotic expansion for Ω smooth and for Ω with "corners". For smooth Ω:

2014	Exner–Minakov–Parnovski	$(3-\varepsilon)$-term asymptotic expansion, 2D
2015	Freitas–Krejčiřík	3-term asymp exp for some domains
2015/6	Pankrashkin–Popoff	3-term, general dim
2017	Kovařík–Pankrashkin	*p*-Laplacian, λ_1
2017	Helffer–Kachmar	*n*-term, general dim
2019	Bögli–K.–Lang	1-term, $\alpha \in \mathbb{C}$

For Ω with model corners or conical:
Helffer–Pankrashkin (2015), Bruneau–Popoff (2016), Pankrashkin (2016), Khalile–Pankrashkin (2018), Khalile (2018), Khalile–Ourmières-Bonafos (2018), Kovařík–Pankrashkin (2019), ...
Also: links to Schrödinger operators with potentials supported on a lower dimensional manifold (δ-potentials), works of Exner and co.; links to magnetic Laplacians, WKB approximations, ...

The dependence of the eigenvalues on Ω

Dirichlet problem

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega \subset \mathbb{R}^d, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

write $0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \ldots$ for the eigenvalues.

Theorem (Faber–Krahn, 1920s/conjecture of Lord Rayleigh)

$\Omega \subset \mathbb{R}^d$ bounded domain, B ball of the same volume. Then

$$\lambda_1(\Omega) \geq \lambda_1(B)$$

with equality iff Ω is a ball (up to a negligible set).

- Analytic version of the *isoperimetric inequality*: $|\partial\Omega| \geq |\partial B|$.
- Physically: circular drums have the lowest fundamental frequencies; "generically speaking", rate of diffusion in a body is slowest if the body is spherical.

The dependence of the Robin eigenvalues on Ω

Now write $\lambda_1(\Omega,\alpha) \leq \lambda_2(\Omega,\alpha) \leq \dots$ for the eigenvalues of the Robin Laplacian on Ω.

Theorem (Bossel–Daners)

$\Omega \subset \mathbb{R}^d$ bounded, sufficiently smooth domain, B ball of the same volume, $\alpha > 0$ fixed.

$$\lambda_1(\Omega) \geq \lambda_1(B)$$

with equality iff Ω is a ball (up to a negligible set).

- Sketch of proof in 2D Bossel (1986), general case Daners (2006), characterisation of equality Daners–K. (2007), less regularity Bucur–Daners (2010), Bucur–Giacomini (2010, 2015)
- Higher eigenvalues, including numerics and/or negative α: various authors since 2008

Divergence from the Weyl asymptotics

Theorem

$\Omega \subset \mathbb{R}^2$ bounded domain of area $|\Omega|$, with $|\partial\Omega|$ sufficiently smooth, $\alpha \in (-\infty, +\infty]$. Then

$$\lambda_n = \frac{4\pi n}{|\Omega|} + \mathcal{O}(n^{1/2}) \qquad \text{as } n \to \infty.$$

Pólya's conjecture

In the Dirichlet case, for all $n \in \mathbb{N}$,

$$\lambda_n(\Omega) \geq \frac{4\pi n}{|\Omega|}.$$

In particular, the minimal values $\inf_{\Omega:|\Omega|=A} \lambda_n(\Omega)$ should also satisfy the Weyl asymptotics.

Observation (Antunes–Freitas–K., 2013)

Let B_n be the disjoint union of n balls of area A/n each and fix $\alpha > 0$. Then

$$\lambda_n(B_n, \alpha) = C(A)n^{1/2} + o(n^{1/2}) \qquad \text{as } n \to \infty.$$

Model case: rectangles (convex domains) and unions of rectangles.

Theorem (Freitas–K., 2019)

Among all rectangles Ω of fixed area, for each fixed $\alpha > 0$

$$\inf_{\Omega} \lambda_n \sim n^{2/3};$$

among all *unions* of rectangles Ω of fixed area,

$$\inf_{\Omega} \lambda_n \sim n^{1/2}.$$

Moreover: for each α, for n large enough the minimiser is always the disjoint union of n equal squares.

For more information (electronic version open access!):

D. Bucur, P. Freitas and J. Kennedy, *Chapter 4: The Robin Problem* in A. Henrot (ed), *Shape optimization and spectral theory*, De Gruyter Open, Warsaw–Berlin, 2017

Muito obrigado pela atenção!

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